1. A The primes between 100 and 120 are 101, 103, 107, 109, 113.

2. A Since 5! divides all three and and no number larger does (since no larger number divides 5!), 5! is the greatest common factor.

3. B Since 121 = 11^2, 121 has 3 divisors (1, 11, 121).

4. D Rearranging, we have \( n^2 - m^2 = 49 \), so \((n-m)(n+m) = 49\). Thus, either \( n-m = n+m = 7 \), \( n+m = 49 \) and \( n-m = 1 \) and \( n+m = 49 \). The first and the last cases yield nonpositive solutions for either \( n \) or \( m \), so they must be excluded. The middle case gives \((m, n) = (24, 25)\).

5. E Letting \( i = a^2 \), \( j = b^3 \), and \( k = c^4 \), it’s apparent that the product \( a^2b^3c^4 \) need not be any integer power of any integer.

6. A \( 2^a \) can only divide \( 3^b \) if it has no factors besides 1 and powers of 3; therefore, \( a = 0 \).

7. B Since \( 89^2 < 20^3 < 90^2 \) and \( 96^2 < 21^3 < 97^2 \), there are \( 96 - 90 + 1 = 7 \) perfect squares between \( 20^3 \) and \( 21^3 \).

8. C Since \( m^2 \) is odd, \( m \) is odd. Let \( m = 2k + 1 \). Thus,

\[
m^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 4n + 3.
\]
Rearranging the final equality and dividing by 2 gives us

\[
2(k^2 + k - n) = 1,
\]
which has no solutions in integers.

9. B 999 and 998 are not prime. 997 is.

10. A The GCM of \( 2^8 \), \( (2^3)(3) \) and \( (2^2)(11) \) is \( (2^8)(3)(11) = 8448 \).

11. D \( 7 \cdot 1 = 7 \), remainder 2 when divided by 5; \( 7 \cdot 2 = 14 \), remainder 4; \( 7 \cdot 3 = 21 \), remainder 1; \( 7 \cdot 4 = 28 \), remainder 3.

12. A Numbers with 9 divisors must be of the form \( p^8 \) or \( p^2q^2 \), where \( p \) and \( q \) are primes. Since \( 2^8 > 200 \), there are no answers of this form. For the other, we have \( 2^2 \cdot 3^2 = 36 \), \( 2^2 \cdot 5^2 = 100 \), \( 2^2 \cdot 7^2 = 196 \). The sum of these is 332.

13. D Since \( 9! = 2^7 \cdot 3^4 \cdot 5 \cdot 7 \), it has \( 8 \cdot 5 \cdot 2 = 160 \) divisors.

14. B Consider the table below, with values of \( n \) across the top and \( m \) along the side and \( 5n + 7m \) in the middle:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>17</td>
<td>22</td>
<td>27</td>
<td>32</td>
<td>37</td>
<td>42</td>
</tr>
<tr>
<td>2</td>
<td>19</td>
<td>24</td>
<td>29</td>
<td>34</td>
<td>39</td>
<td>44</td>
<td>49</td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>31</td>
<td>36</td>
<td>41</td>
<td>46</td>
<td>51</td>
<td>56</td>
</tr>
<tr>
<td>4</td>
<td>33</td>
<td>38</td>
<td>43</td>
<td>48</td>
<td>53</td>
<td>58</td>
<td>63</td>
</tr>
<tr>
<td>5</td>
<td>40</td>
<td>45</td>
<td>50</td>
<td>55</td>
<td>60</td>
<td>65</td>
<td>70</td>
</tr>
</tbody>
</table>
Thus, we can form any number \( k \) such that \( k \equiv 2 \pmod{5} \) and \( k \geq 12 \), any \( k \equiv 4 \pmod{5} \) and \( k \geq 19 \), any \( k \equiv 1 \pmod{5} \) and \( k \geq 26 \), \( k \equiv 3 \pmod{5} \) and \( k \geq 33 \), any \( k \equiv 0 \pmod{5} \) and \( k \geq 40 \). Any number in these congruence classes below the stated minimum cannot be formed. Thus, 35 is the largest that cannot be formed. (Note that the above sketches out a general proof that the largest number that cannot be written in the form \( km + jn \) for positive \( m, n \) is \( k \) if \( (k, j) = 1 \), and that the largest with nonnegative \( m, n \) is \( k j - k - j \).)

15. C  Simplifying each gives us:

\[
x = 2^{22^2} = 2^{16}, y = 3^{33^3} = 3^{27}, z = 4^{4^4} = 4^{2^8} = (2^2)^{2^8} = 2^{2^9}.
\]

Thus, obviously \( y > x \) since \( 27 > 16 \) and \( 3 > 2 \), and \( x > z \) because \( 2^{16} > 2^9 \). Thus, \( z < x < y \).

16. E  Since \( 20 = 2^2 \cdot 5 \), we must find how many factors of 5 and of 2 are in \( 2002! \). There’s a factor of 5 for every multiple of 5 in the product \( 2002! \), another for every multiple of 25, another for every multiple of 125, and so on, for a total of:

\[
\left\lfloor \frac{2002}{5} \right\rfloor + \left\lfloor \frac{2002}{25} \right\rfloor + \left\lfloor \frac{2002}{125} \right\rfloor + \left\lfloor \frac{2002}{625} \right\rfloor = 499.
\]

By the same method, we find that there are 1995 factors of 2 in \( 2002! \), so \( 2^2 \) divides it \( \left\lfloor \frac{1995}{2} \right\rfloor = 997 \) times. Thus, 5 is the limiting factor, so \( n = 499 \) is the largest permissible \( n \).

17. C  Consider the equation \( \pmod{9} \). Since \( 63n^3 \equiv 0 \pmod{9} \) and \( 777700006 \equiv 7 \pmod{9} \), we must find an \( m \) such that \( m^3 \equiv 7 \pmod{9} \). However, if we write \( m = 3k + j \), where \( j = 0, 1, \) or \( 2 \), then

\[
m^3 = (3k + j)^3 = 27k^3 + 27k^2j + 9kj^2 + j^3.
\]

The first three terms are divisible by 9, so \( m^3 \equiv j^3 \pmod{9} \). Since \( j = 0, 1, \) or \( 2 \), \( m^3 \equiv 0, 1, \) or \( 8 \pmod{9} \) for all \( m \). Thus, there are no solutions to the equation.

18. A  Consider two integers \( 0 \leq k < j < 103 \). We aim to prove that \( j^2 \equiv k^2 \pmod{103} \) iff \( j + k = 103 \).

If \( j^2 \equiv k^2 \pmod{103} \), then \( j^2 - k^2 = 103m \) for some integer \( m \). Thus, \( (j - k)(j + k) = 103m \).

Hence, either \( j - k \) or \( j + k \) is divisible by 103. Since \( j \neq k \) and \( 0 \leq k < j < 103 \), \( 0 < j - k < 103 \) and \( 0 < j + k < 2 \cdot 103 \), so \( j + k \) must equal 103. Similarly, we can show that any \( j, k \) such that \( j + k = 103 \) satisfies \( j^2 \equiv k^2 \pmod{103} \). Hence, no two of the squares \( 0^2, 1^2, 2^2, \ldots, 51^2 \) are congruent mod 103, and all the squares from 52^2 through 103^2 are congruent to a square in the first list. Any squares of larger integers are clearly congruent to one of the squares from \( 0^2 \) to \( 103^2 \), so they need not be considered. Thus, there are 52 different quadratic residues of 103.

19. C  In algebraic terms, we have \( 6 \cdot 10^n + k = 25 \cdot k \), or \( 5^n \cdot 2^{n-2} = k \). Hence, our solutions are \( (n = 2,3,4,5): 625,6250,62500,625000 \).

20. B  We seek integers \( n \) such that \( 1000 < n^2 < 10000 \) and all digits of \( n^2 \) are even. First, \( n \) cannot be odd. Suppose \( n \) ends in 0. Quick examination yields \( n = 80 \) as the only solution of this form. Suppose \( n = 10a + 4 \). Thus, \( n^2 = 100a^2 + 40a + 16 \), hence the tens digit will always be odd. Similarly, we can dismiss \( n \) of the form \( 10a + 6 \). This leaves us \( n \) that end in 2 or 8. Inspection reveals solutions \( n = 92, 68, 78 \), for a total of 4 solutions.
21. A Rearranging yields $10(m + n) = mn$, or 

$$(m - 10)(n - 10) = 100.$$ 

Since 100 has 9 divisors, there are 9 values $m - 10$ can take on (with $n - 10$ equal to $100/(m - 10)$). Moreover, $m - 10$ and $n - 10$ could both be negative, yielding another 9 solutions. However, we must omit the solution $m = n = 0$ since that would give us $1/0 + 1/0 = 1/10$ as our initial problem. Thus, there are 17 solutions.

22. C Since $7^1 = 7$, $7^2 = 49$, $7^3 = 343$, and $7^4 = 2401$, the last two digits of successive powers of 7 go in a cycle: 07, 49, 43, 01. Thus, the problem is reduced to evaluating $7^7$ (mod 4). Since $7 \equiv 3$ (mod 4), the powers of 7 alternate 3, 1, 3, 1, ... (mod 4). Hence, $7^7 \equiv 3$ (mod 4), so $7^7$ (mod 100) and the tens digit is therefore 4.

23. B Let there be $k$ 9th graders, so there are $10 \cdot k$ 10th graders and $n = 11k$ total students. The 9th graders play each other in $\binom{k}{2} = \frac{k(k - 1)}{2}$ ways; no matter how these games turn out, they will collectively earn $\frac{k(k - 1)}{2}$ points. Similarly, the 10th graders get $\binom{10k}{2} = 5k(10k - 1)$ points from their games amongst themselves.

Now suppose that the 10th graders get a total of $j$ points from the $10k^2$ games between 9th and 10th graders, leaving the 9th graders $10k^2 - j$ points from those games. Since the 10th graders have a total of 4.5 times as many points as the 9th graders, we have:

$$5k(10k - 1) + j = 4.5\left(\frac{k(k - 1)}{2}\right) + 10k^2 - j$$

$$\therefore 20k^2 - 20k + 4j = 9k^2 - 9k + 180k^2 - 18j$$

$$\therefore 2j = k - k^2.$$ 

This final equation only has one solution for $j \geq 0, k > 0$, namely $(j, k) = (0, 1)$. Thus, the only possible value of $n$ is 11.

24. B Clearly no $m$ which are multiples of 3 can have powers which are only 1 more than a multiple of $3^{10}$. For other $m$, we use Euler’s generalization of Fermat’s Theorem since $(m, 3^{10}) = 1$:

$$m^{\phi(3^{10})} \equiv 1 \pmod{3^{10}}$$

Since $\phi(3^{10}) = 3^{10}(1 - 1/3) = 3^{10} - 3^9$, we have

$$m^{3^{10} - 3^9} \equiv 1 \pmod{3^{10}}$$

for all $m < 1000$ which are not divisible by three. There are $2/3(999) = 666$ such integers.

25. D Let $S(n)$ be the sum of the digits of $n$. Thus, we seek $S(S(S(4444^{4444})))$. Since $4444^{4444} < 10000^{5000}$, and $10000^{5000}$ is 1 followed by 4 · 5000 = 20000 zeroes,

$$S(4444^{4444}) < S(999 \ldots 999) = 9 \times 20000 = 180000,$$

. Since $S(4444^{4444}) < 180000$, $S(S(4444^{4444})) < S(99999) = 45$ because 99999 has the largest sum of digits of numbers less than 180000. Finally, $S(S(S(4444^{4444}))) < S(39) = 12$ because 39 has the
largest sum of digits among numbers less than 45. Hence our answer is less than 12. To find the answer, we observe that $S(n) \equiv n \pmod{9}$ for all $n$, (prove this by noting that $10 \equiv 1 \pmod{9}$). Hence,

$$S(S(S(4444^{4444}))) \equiv 4444^{4444} \pmod{9} \equiv 7^{4444}.$$ 

The powers of 7 cycle 7, 4, 1, 7, 4, 1... (mod 9) and 4444 \equiv 1 \pmod{3}, so $7^{4444} \equiv 7 \pmod{9}$. Since 7 is the only positive number less than 12 which is congruent to 7 mod 9,

$$S(S(S(4444^{4444}))) = 7.$$