Solutions
2002 Sequences and Series Topic Test – Mu Division

1. Evaluate the sum \( \sum_{k=1}^{n} k(k+1) \).

Solution: Using the sum of the first \( n \) integers and the sum of the squares of the first \( n \) integers.

\[
\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{(n)(n+1)(2n+1)}{6} + \frac{(n)(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.
\]

(A) \( \frac{n(n^2+1)}{3} \)  [(B) \( \frac{n(n+1)(n+2)}{3} \)]  (C) \( \frac{2n(n^2-1)}{3} \)  (D) \( \frac{n^2(n+1)}{2} \)  (E) NOTA

2. Evaluate the infinite series \( \sum_{k=1}^{\infty} \left( \frac{k}{k+1} - \frac{k+1}{k+2} \right) \).

Solution: Telescoping sums is the trick here.

Let \( S_n = \sum_{k=1}^{n} \left( \frac{k}{k+1} - \frac{k+1}{k+2} \right) = \frac{1}{2} - \frac{n+1}{n+2} \rightarrow -\frac{1}{2} \) as \( n \rightarrow \infty \).

(A) divergent  (B) \( \frac{1}{2} \)  (C) 0  [(D) \( \frac{1}{2} \)]  (E) NOTA

3. Evaluate the product \( \prod_{k=2}^{n} \left( 1 - \frac{1}{k^2} \right) \).

Solution: Experimental mathematics, i.e., try \( n = 2,3,4 \) and it should become clear that (B) is a good candidate for the product. Mathematical induction will show this as well considering

\[
\log \left( \prod_{k=2}^{n} \left( 1 - \frac{1}{k^2} \right) \right) = \sum_{k=2}^{n} \log \left( 1 - \frac{1}{k^2} \right)
\]

\[
= \sum_{k=2}^{n} \log \left( \frac{k^2 - 1}{k^2} \right)
\]

\[
= \sum_{k=2}^{n} (-2 \log(k) + \log(k+1) + \log(k-1)) = -\log(n) + \log(n+1) - \log(2)
\]

\[
= \log \left( \frac{n+1}{2n} \right).
\]

(A) \( \frac{2n-1}{2n} \)  [(B) \( \frac{n+1}{2n} \)]  (C) \( \frac{2n-1}{n^2} \)  (D) \( \frac{n^2-1}{4} \)  (E) NOTA
4. Find the limit of \( s_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \) as \( n \to \infty \). \( n = 1,2,3,\cdots \).

Solution: Use sum of first \( n \) integers and algebra.

\[
s_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2} \left[ 1 + \frac{1}{n} \right] \to \frac{1}{2} \text{ as } n \to \infty.
\]

(A) 2  (B) 0  (C) diverges  [(D) \( \frac{1}{2} \)]  (E) NOTA

5. Find the limit of \( s_n = 1 - \frac{1}{2} + \frac{1}{4} - \cdots + (-\frac{1}{2})^n \) as \( n \to \infty \). \( n = 1,2,3,\cdots \).

Solution: Geometric series with \( r = -\frac{1}{2} \). Thus, \( s_n = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3} \).

[(A) \( \frac{2}{3} \)]  (B) 2  (C) \( \frac{1}{3} \)  (D) \( \frac{1}{2} \)  (E) NOTA

6. Evaluate the limit, \( \lim_{n \to \infty} \sqrt{n} \left( \sqrt{n+1} - \sqrt{n} \right) \).

Solution: Multiple numerator and denominator by \( \left( \sqrt{n+1} + \sqrt{n} \right) \).

\[
\sqrt{n} \left( \sqrt{n+1} - \sqrt{n} \right) = \frac{\sqrt{n+1} - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n}} + \sqrt{1}} \to \frac{1}{2} \text{ as } n \to \infty.
\]

(A) diverges  (B) 2  (C) 0  [(D) \( \frac{1}{2} \)]  (E) NOTA

7. Evaluate: \( \lim_{n \to \infty} \frac{1 - \left(1 - \frac{1}{n} \right)^3}{1 - \left(1 - \frac{1}{n} \right)} \).

Solution: Note that \( \frac{1-a^3}{1-a} = 1 + a + a^2 \). Thus,

\[
\lim_{n \to \infty} \frac{1 - \left(1 - \frac{1}{n} \right)^3}{1 - \left(1 - \frac{1}{n} \right)} = \lim_{n \to \infty} 1 + \left(1 - \frac{1}{n} \right) + \left(1 - \frac{1}{n} \right)^2 = 3.
\]

(A) 0  (B) undefined  (C) \( \frac{1}{3} \)  [(D) 3]  (E) NOTA
8. The series \( \sum_{n=0}^{\infty} (k^2 - 3)^n \) converges for which values of \( k \)?

Solution: This is a geometric series with \( r = k^2 - 3 \). The series converges for all values of \( k \) for which \( |k^2 - 3| < 1 \). This is equivalent to \(-1 < k^2 - 3 < 1\) or \( 2 < k^2 < 4 \). Taking square roots gives \(-2 < k < -\sqrt{2} \) or \( \sqrt{2} < k < 2 \)

(A) \(-1 < k < 1\)  (B) \(-2 < k < -\sqrt{2} \) or \( \sqrt{2} < k < 2 \)  (C) \(-\sqrt{2} < k < \sqrt{2} \)

(D) \( k < -2 \) or \( k > 2 \)  (E) NOTA

9. The Fibonacci sequence satisfies the recurrence relation \( F_k = F_{k-1} + F_{k-2} \), for all integers \( k \geq 2 \), with \( F_0 = 1 \) and \( F_1 = 1 \). Evaluate the limit \( \lim_{k\to\infty} \frac{F_{k+1}}{F_k} \), assuming this limit exists.

Solution: Let \( L = \lim_{k\to\infty} \frac{F_{k+1}}{F_k} \) and recognize that also \( L = \lim_{k\to\infty} \frac{F_k}{F_{k-1}} \). Then,

\[
L = \lim_{k\to\infty} \frac{F_{k+1}}{F_k} = \lim_{k\to\infty} \frac{F_k + F_{k-1}}{F_k} = \lim_{k\to\infty} 1 + \frac{F_{k-1}}{F_k} = 1 + \frac{1}{L}.
\]

Now solving for \( L \) gives

\[
L = \frac{1 + \sqrt{5}}{2}.
\]

Since the terms of this sequence are positive \( L = \frac{1 + \sqrt{5}}{2} \).

(A) \( \frac{1 - \sqrt{5}}{2} \)  (B) 1  (C) \( \frac{\sqrt{5}}{2} \)  (D) \( \frac{1 + \sqrt{5}}{2} \)  (E) NOTA

10. Which fraction represents the repeating decimal \( 0.321321\ldots \)?

Solution: Another geometric series problem.

\[
0.321321\ldots = \sum_{k=0}^{\infty} .321 \left( \frac{1}{1000} \right)^k = .321 \left[ \frac{1}{1 - \frac{1}{1000}} \right] = .321 \frac{1000}{999} = \frac{321}{999}.
\]

(A) \( \frac{999}{321} \)  (B) \( \frac{321}{99} \)  (C) \( \frac{321}{999} \)  (D) \( \frac{1000}{321} \)  (E) NOTA
11. Evaluate: \( \sum_{k=1}^{\infty} \frac{(-x)^k}{k!} \).

Solution: The power series above is that of \( e^{-x} - 1 \). Note that the \( k = 0 \) term is missing from this power series.

(A) \( e^x \)  (B) \( e^{-x} \)  (C) \( 1 + e^x \)  (D) \( e^x + 1 \)  [E) NOTA]

12. Which of the following is true of the series \( \sum_{n=1}^{\infty} \frac{(-1)^n}{1 + 5n} \)?

Solution: By the alternating series test this is a convergent series, but the series formed by taking the absolute value of each of its terms give a divergent series. Hence it is a conditionally convergent series.

(A) absolutely convergent  (B) divergent  [(C) conditionally convergent]

(D) almost convergent  (E) NOTA

13. Evaluate: \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-2}}{(2n-2)!} \).

Solution: Note the the Maclaurin expansion for \( \cos(x) \) can be written as

\[
\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{(2n-2)!}, \quad \text{so} \quad \cos(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-2}}{(2n-2)!}.
\]

(A) \( \cos(\sqrt{2}) \)  (B) \( \sin(\sqrt{2}) \)  [(C) \( \cos(2) \)]  (D) \( \ln(2) \)  (E) NOTA

14. If \( \sum_{j=0}^{\infty} b_j \) is a convergent series of nonnegative terms and there are constants \( M \) and \( J \) such that \( |a_j| \leq Mb_j \) for \( j \geq J \), then which of the following statements describes the convergence of the series \( \sum_{j=0}^{\infty} a_j \) :

Solution: From the given hypothesis we have that the series \( \sum_{j=0}^{\infty} a_j \) is absolutely convergent by the comparison test.

(A) conditionally convergent  (B) uniformly convergent  [(C) absolutely convergent ]  (D) divergent  (E) NOTA
15. Evaluate: $\sum_{k=3}^{\infty} \left[ \sin \left( \frac{4}{k} \right) - \sin \left( \frac{4}{k+2} \right) \right]$. 

Solution: Another telescoping problem, but an interesting expansion:

$$\sum_{k=3}^{n} \left[ \sin \left( \frac{4}{k} \right) - \sin \left( \frac{4}{k+2} \right) \right] = \sin \left( \frac{4}{3} \right) + \sin(1) - \sin \left( \frac{4}{n+1} \right) - \sin \left( \frac{4}{n+2} \right)$$

$\rightarrow \sin \left( \frac{4}{3} \right) + \sin(1) \text{ as } n \rightarrow \infty.$

(A) $\sin \left( \frac{4}{3} \right)$ (B) 0 [(C) $\sin \left( \frac{4}{3} \right) + \sin(1)$] (D) $\sin(1)$ (E) NOTA

16. Evaluate the limit $\lim_{n \to \infty} \left( \frac{\arctan(n^2)}{n^2 + 1} \right)^n$.

Solution: Since $|\arctan(n^2)| \leq \frac{\pi}{2}$ then the ratio $\frac{\arctan(n^2)}{n^2 + 1} \rightarrow 0$ as $n \rightarrow \infty$.

Hence the $n^{th}$ power of this ratio also has limit 0.

(A) $\pi$ [(B) 0] (C) $\frac{\pi}{2}$ (D) $e$ (E) NOTA

17. Evaluate: $\sum_{j=1}^{\infty} \frac{1}{j(j+1)}$.

Solution: Telescoping series with $n^{th}$ partial sum given by

$$S_n = \sum_{j=1}^{n} \frac{1}{j(j+1)} = \sum_{j=1}^{n} \left[ \frac{1}{j} - \frac{1}{j+1} \right] = 1 - \frac{1}{n+1} \rightarrow 1 \text{ as } n \rightarrow \infty.$$ 

(A) 2 [(B) 1] (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) NOTA

18. The $10^{th}$ term of an arithmetic sequence is 52 and the $15^{th}$ is 77. Find the $50^{th}$ term of this sequence.

Solution: Letting the $n^{th}$ term $s_n = a + nb$ we have the system of equations $a + 10b = 52$ and $a + 15b = 77$. Solving this system for $a$ and $b$ give $s_n = 2 + 5n$. So the $50^{th}$ term is $s_{50} = 2 + 50 \cdot 5 = 252$.

[(A) 252] (B) 250 (C) 302 (D) –48 (E) NOTA
19. The Maclaurin series for $e^x + e^{3x}$ is

Solution: Adding the Maclaurin series for $e^x$ and $e^{3x}$ gives

$$e^x + e^{3x} = \sum_{j=0}^{\infty} \frac{x^j}{j!} + \sum_{j=0}^{\infty} \frac{(3x)^j}{j!} = \sum_{j=0}^{\infty} \frac{(1 + 3)^j}{j!} x^j.$$  

(A) $\sum_{j=0}^{\infty} \frac{x^j}{j!} j^3$  [ B) $\sum_{j=0}^{\infty} \frac{(1 + 3)^j}{j!} x^j$ ]  (C) $\sum_{j=0}^{\infty} \frac{j!}{j} x^j$  (D) $\sum_{j=0}^{\infty} \frac{(1 + 3)^j}{j!} x^{j-1}$  

(E) NOTA

20. Evaluating the improper integral $\int_{b=\pi}^{b=\pi} \frac{1}{x \ln(x)} dx$ shows that $\sum_{j=2}^{\infty} \frac{(-1)^j}{j \ln(j)}$ 

Solution: Since the improper integral does not exist, we have that the series is not absolutely convergent, but this is not one of the choices (A)-(D) so it is (E).

(A) sums to $\ln(\ln(2))$.  (B) converges absolutely.  (C) diverges.  

(D) converges conditionally.  [(E) NOTA]

21. Applying the alternating series test to $\sum_{j=2}^{\infty} \frac{(-1)^j}{j \ln j}$ shows this series 

Solution: The $n^{th}$ of this series goes to 0 and it alternates in sign, so it is a convergent series.

(A) converges absolutely.  [(B) converges.]  (C) diverges.  

(D) does not converge absolutely.  (E) NOTA.

22. Evaluate $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Solution: (E) since we are given a power series in the variable $x$ and (A) through (D) are constants.

(A) $e$  (B) $e^2$  (C) $2e$  (D) $e^{-2}$  [(E) NOTA]
23. Evaluate \( \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n2^n} \).

Solution: This is the Taylor series for \( \ln(1 + x) \) evaluated at \( x = -\frac{1}{2} \). Note that

\[
\ln\left(\frac{1}{2}\right) = -\ln(2) \quad \text{and} \quad \ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n,
\]

so letting \( x = -\frac{1}{2} \) gives

\[
-\ln(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n \cdot 2^n}.
\]

(A) \( \ln(2) \)  [(B) \( -\ln(2) \)]  (C) 0  (D) \( \ln(3) \)  (E) NOTA

24. Which of the following describes the convergence of \( \sum_{k=0}^{\infty} \frac{(-2)^k}{k^3} \).

Solution: By the ratio test this series diverges. Consider,

\[
\left| \frac{a_{k+1}}{a_k} \right| = \left| \frac{(-2)^{k+1}}{(k+1)^3} \right| = 2 \left( \frac{k}{k+1} \right)^3 \to 2 \quad \text{as} \quad k \to \infty.
\]

Since this limit is greater than 1 then by the ratio test the series is divergent.

(A) converges  (B) converges absolutely  (C) converges conditionally

[ (D) diverges]  (E) NOTA

25. Evaluate \( \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} \).

Solution: The Maclaurin expansion for \( \sin(x) \) is given by

\[
\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},
\]

thus the sum of the series is \( \sin(1) \).

(A) 0  [(B) \( \sin(1) \)]  (C) \( \cos(1) \)  (D) diverges  (E) NOTA
26. The interval of convergence of \( \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{x}{3} \right)^k \) is

Solution: For \( a_k = \frac{1}{k} \left( \frac{x}{3} \right)^k \), consider the ratio

\[
\frac{a_{k+1}}{a_k} = \frac{1}{1 + \frac{x}{3}} \rightarrow \frac{|x|}{3} \text{ as } k \to \infty.
\]

Thus, \( \left| \frac{x}{3} \right| < 1 \), or when \( |x| < 3 \). Now we only have to check the end points. At \( x = 3 \) this series reduces to the harmonic series which is divergent. At \( x = -3 \) it reduces to the alternating series \( \sum_{k=1}^{\infty} (-1)^k \frac{x}{k} \) which is convergent by the alternating series test.

Thus, \([-3,3)\) is the interval of convergence.

[(A) \([-3,3)\)]  (B) \([-\frac{1}{3}, \frac{1}{3}]\)  (C) \((-3,3]\)  (D) \([-1,1]\)  (E) NOTA

27. What are the terms up to degree 4 in the Maclaurin series of \( \frac{\sin(x)}{1-x} \)?

Solution: Either divide \( \frac{\sin(x)}{1-x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots}{1-x} \) which up to 4th order gives terms \( x + x^2 + \frac{5}{6} x^3 + \frac{5}{6} x^4 \) or multiple the Maclaurin expansions of \( \frac{1}{1-x} \) and \( \sin(x) \), \( 1 + x + x^2 + x^3 + \cdots \left( x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right) \) which in either case gives \( \frac{\sin(x)}{1-x} \) as \( x + x^2 + \frac{5}{6} x^3 + \frac{5}{6} x^4 \) up to terms of order four.

(A) \( x - x^2 - \frac{7}{6} x^3 - \frac{5}{6} x^4 \)  (B) \( x - x^2 - \frac{x^3}{6} + \frac{x^4}{6} \)  ([C) \( x + x^2 + \frac{5}{6} x^3 + \frac{5}{6} x^4 \)]

(D) \( x + x^2 + \frac{x^3}{6} + \frac{x^4}{6} \)  (E) NOTA
28. At the first of each month, for ten years, $1,000 is deposited into a saving account earning 6% a year compounded monthly. How much money is in this account when the last deposit is made? Round to nearest dollar.

Solution: Here $S_n$ is the amount in the account at the start of each month.

\[ \begin{align*}
S_1 &= 1000. \\
S_2 &= 1000 \left(1 + \frac{0.06}{12}\right) + 1000. \\
S_3 &= 1000 \left(1.005\right) + 1000[1.005 + 1000 \\
&= 1000(1.005)^2 + 1.005 + 1]. \\
& \vdots \\
S_{120} &= \sum_{k=0}^{120} 1000(1.005)^{k-1} \\
&= \frac{1000[(1.005)^{120} - 1]}{1.005 - 1} \\
& \approx 163,879.
\end{align*} \]

(A) $6,958,240$  (B) $165,699$  (C) $48,000$  [(D) $163,879$]  (E) NOTA

29. Which is true of the series \[\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}?\]

Solution: This is a p-series with $p=3/2 > 1$ and hence a convergent series.

[(A) it converges.]  (B) its sum is \( \frac{3}{\pi} \).  (C) it is conditionally convergent  
(D) it is divergent.  (E) NOTA

30. Which is true for the sequence \( \left\{ \frac{5}{3 + (-1)^n} \right\} \)?

Solution: This sequence oscillates between the values 5/4 and 5/2 and hence it is divergent by oscillation.

(A) it converges to \( \frac{5}{3} \).  (B) it is unbounded. [(C) it is divergent by oscillation]  
(D) it converges to \( \frac{15}{4} \)  (E) NOTA
TIEBREAKER:
Solution: First note that the $\nu^{th}$ term of this series is given by

$$a_\nu = a^{\ln(\nu)}$$

$$= e^{\ln(a)\ln(\nu)}$$

$$= e^{\ln(\nu)\ln(a)}$$

$$= \nu^{\ln(a)}.$$

Thus, this is a $p$-series with $p = \ln(a)$ and it converges when $p = \ln(a) < -1$. This is true for all $a < e^{-1}$. 