Solutions

2002 Sequences and Series Topic Test – Mu Division

1. Evaluate the sum $\sum_{k=1}^{n} k(k+1)$.

Solution: Using the sum of the first n integers and the sum of the squares of the first n integers.

$$\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{(n)(n+1)(2n+1)}{6} + \frac{(n)(n+1)}{2} = \frac{n(n+1)(n+2)}{3}.$$

(A) $\frac{n(n^2+1)}{3}$ [(B) $\frac{n(n+1)(n+2)}{3}$] (C) $\frac{2n(n^2-1)}{3}$ (D) $\frac{n^2(n+1)}{2}$ (E) NOTA

2. Evaluate the infinite series $\sum_{k=1}^{\infty}(\frac{k}{k+1} - \frac{k+1}{k+2})$.

Solution: Telescoping sums is the trick here.

Let $S_n = \sum_{k=1}^{n}\left(\frac{k}{k+1} - \frac{k+1}{K+2}\right) = \frac{1}{2} - \frac{n+1}{n+2} \to -\frac{1}{2}$ as $n \to \infty$.

(A) divergent (B) $\frac{1}{2}$ (C) 0 [(D) $-\frac{1}{2}$] (E) NOTA

3. Evaluate the product $\prod_{k=2}^{n}(1 - \frac{1}{k^2})$.

Solution: Experimental mathematics, i.e., try $n = 2,3,4$ and it should become clear that (B) is a good candidate for the product. Mathematical induction will show this as will considering

$$\log\left(.\prod_{\mathrm{k}=2}^{n}\left(1 - \frac{1}{\mathrm{k}^2}\right)\right) = \sum_{k=2}^{n}\log\left(1 - \frac{1}{k^2}\right)$$

$$= \sum_{k=2}^{n}\log\left(\frac{k^2-1}{k^2}\right)$$

$$= \sum_{k=2}^{n}\left(-2\log(k) + \log(k+1) + \log(k-1)\right)$$

$$= -\log(n) + \log(n+1) - \log(2)$$

$$= \log\left(\frac{n+1}{2n}\right).$$

(A) $\frac{2n-1}{2n}$ [(B) $\frac{n+1}{2n}$] (C) $\frac{2n-1}{n^2}$ (D) $\frac{n^2-1}{4}$ (E) NOTA

4. Find the limit of $s_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$ as $n \to \infty$. $n = 1,2,3,\cdots$.

Solution: Use sum of first n integers and algebra.

$$s_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} = \sum_{k=1}^{n} \frac{k}{n^2} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2} \cdot \frac{n(n+1)}{2} = \frac{1}{2}\left[1 + \frac{1}{n}\right] \to \frac{1}{2} \text{ as } n \to \infty.$$

(A) 2 (B) 0 (C) diverges [(D) $\frac{1}{2}$] (E) NOTA

5. Find the limit of $s_n = 1 - \frac{1}{2} + \frac{1}{4} - \cdots + (-\frac{1}{2})^n$ as $n \to \infty$. $n = 1,2,3,\cdots$.

Solution: Geometric series with $r = -\frac{1}{2}$. Thus, $s_n = \dfrac{1}{1 - \left(-\frac{1}{2}\right)} = \frac{2}{3}$.

[(A) $\frac{2}{3}$] (B) 2 (C) $\frac{1}{3}$ (D) $\frac{1}{2}$ (E) NOTA

6. Evaluate the limit, $\lim_{n\to\infty} \sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right)$

Solution: Multiple numerator and denominator by $\left(\sqrt{n+1} + \sqrt{n}\right)$.

$$\sqrt{n}\left(\sqrt{n+1} - \sqrt{n}\right) = \frac{\sqrt{n}(n+1-n)}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \to \frac{1}{2} \text{ as } n \to \infty.$$

(A) diverges (B) 2 (C) 0 [(D) $\frac{1}{2}$] (E) NOTA

7. Evaluate: $\lim_{n\to\infty} \dfrac{1 - \left(1 - \frac{1}{n}\right)^3}{1 - \left(1 - \frac{1}{n}\right)}$.

Solution: Note that $\frac{1 - a^3}{1 - a} = 1 + a + a^2$. Thus,

$$\lim_{n\to\infty} \frac{1 - \left(1 - \frac{1}{n}\right)^3}{1 - \left(1 - \frac{1}{n}\right)} = \lim_{n\to\infty} 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)^2 = 3.$$

(A) 0 (B) undefined (C) $\frac{1}{3}$ [(D) 3] (E) NOTA

8. The series $\sum_{n=0}^{\infty}\left(k^2-3\right)^n$ converges for which values of k?

Solution: This is a geometric series with $r = k^2 - 3$. The series converges for all values of k for which $\left|k^2-3\right| < 1$. This is equivalent to $-1 < k^2 - 3 < 1$ or $2 < k^2 < 4$. Taking square roots gives $-2 < k < -\sqrt{2}$ or $\sqrt{2} < k < 2$

(A) -1< k < 1 [(B) $-2 < k < -\sqrt{2}$ or $\sqrt{2} < k < 2$] (C) $-\sqrt{2} < k < \sqrt{2}$

(D) $k < -2$ or $k > 2$ (E) NOTA

9. The Fibonacci sequence satisfies the recurrence relation $F_k = F_{k-1} + F_{k-2}$, for all integers $k \geq 2$, with $F_0 = 1$ and $F_1 = 1$. Evaluate the $\lim_{k\to\infty}\frac{F_{k+1}}{F_k}$, assuming this limit exits.

Solution: Let $L = \lim_{k\to\infty}\frac{F_{k+1}}{F_k}$ and recognize that also $L = \lim_{k\to\infty}\frac{F_k}{F_{k-1}}$. Then,

$L = \lim_{k\to\infty}\frac{F_{k+1}}{F_k} = \lim_{k\to\infty}\frac{F_k + F_{k-1}}{F_k} = \lim_{k\to\infty} 1 + \frac{F_{k-1}}{F_k} = 1 + \frac{1}{L}$. Now solving for L gives $L = \frac{1 \pm \sqrt{5}}{2}$. Since the terms of this sequences are positive $L = \frac{1+\sqrt{5}}{2}$.

(A) $\frac{1-\sqrt{5}}{2}$ (B) 1 (C) $\frac{\sqrt{5}}{2}$ [(D) $\frac{1+\sqrt{5}}{2}$] (E) NOTA

10. Which fraction represents the repeating decimal $0.321321\ldots$?

Solution: Another geometric series problem.

$$0.321321\cdots = \sum_{k=0}^{\infty} .321\left(\frac{1}{1000}\right)^k = .321\left[\frac{1}{1-\frac{1}{1000}}\right] = .321\frac{1000}{999} = \frac{321}{999}.$$

(A) $\frac{999}{321}$ (B) $\frac{321}{99}$ [(C) $\frac{321}{999}$] (D) $\frac{1000}{321}$ (E) NOTA

11. Evaluate: $\sum_{k=1}^{\infty} \frac{(-x)^k}{k!}$..

Solution: The power series above is that of $e^{-x} - 1$. Note that the $k = 0$ term is missing from this power series.

(A) e^x (B) e^{-x} (C) $1 + e^x$ (D) $e^x + 1$ [(E) NOTA]

12. Which of the following is true of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{1+5n}$?

Solution: By the alternating series test this is a convergent series, but the series formed by taking the absolute value of each of its terms give a divergent series. Hence it is a conditionally convergent series.

(A) absolutely convergent (B) divergent [(C) conditionally convergent]

(D) almost convergent (E) NOTA

13. Evaluate: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-2}}{(2n-2)!}$.

Solution: Note the the Maclaurin expansion for $\cos(x)$ can be written as

$\cos(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-2}}{(2n-2)!}$, so $\cos(2) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2^{2n-2}}{(2n-2)!}$.

(A) $\cos(\sqrt{2})$ (B) $\sin(\sqrt{2})$ [(C) $\cos(2)$] (D) $\ln(2)$ (E) NOTA

14. If $\sum_{j=0}^{\infty} b_j$ is a convergent series of nonnegative terms and there are constants M and J such that $|a_j| \le M b_j$ for $j \ge J$, then which of the following statements describes the convergence of the series $\sum_{j=0}^{\infty} a_j$:

Solution: From the given hypothesis we have that the series $\sum_{j=0}^{\infty} a_j$ is absolutely convergent by the comparison test.

(A) conditionally convergent (B) uniformly convergent [(C) absolutely convergent] (D) divergent (E) NOTA

15. Evaluate: $\sum_{k=3}^{\infty}\left[\sin\left(\frac{4}{k}\right)-\sin\left(\frac{4}{k+2}\right)\right]$.

Solution: Another telescoping problem, but an interesting expansion:

$$\sum_{k=3}^{n}\left[\sin\left(\frac{4}{k}\right)-\sin\left(\frac{4}{k+2}\right)\right]=\sin\left(\frac{4}{3}\right)+\sin(1)-\sin\left(\frac{4}{n+1}\right)-\sin\left(\frac{4}{n+2}\right)$$

$$\to \sin\left(\frac{4}{3}\right)+\sin(1) \text{ as } n\to\infty.$$

(A) $\sin\left(\frac{4}{3}\right)$ (B) 0 [(C) $\sin\left(\frac{4}{3}\right)+\sin(1)$] (D) $\sin(1)$ (E) NOTA

16. Evaluate the limit $\lim_{n\to\infty}\left\{\frac{\arctan(n^2)}{n^2+1}\right\}^n$.

Solution: Since $|\arctan(n^2)|\le\frac{\pi}{2}$ then the ratio $\frac{\arctan(n^2)}{n^2+1}\to 0$ as $n\to\infty$.
Hence the n^{th} power of this ratio also has limit 0.

(A) π [(B) 0] (C) $\frac{\pi}{2}$ (D) e (E) NOTA

17. Evaluate: $\sum_{j=1}^{\infty}\frac{1}{j(j+1)}$.

Solution: Telescoping series with n^{th} partial sum given by

$$S_n=\sum_{j=1}^{n}\frac{1}{j(j+1)}=\sum_{j=1}^{n}[\frac{1}{j}-\frac{1}{j+1}]=1-\frac{1}{n+1}\to 1 \text{ as } n\to\infty.$$

(A) 2 [(B) 1] (C) $\frac{1}{2}$ (D) $\frac{2}{3}$ (E) NOTA

18. The 10^{th} term of an arithmetic sequence is 52 and the 15^{th} is 77. Find the 50^{th} term of this sequence.

Solution: Letting the n^{th} term $s_n=a+nb$ we have the system of equations $a+10b=52$ and $a+15b=77$. Solving this system for a and b give $s_n=2+5n$. So the 50^{th} term is $s_{50}=2+50\cdot 5=252$.

[(A) 252] (B) 250 (C) 302 (D) –48 (E) NOTA

19. The Maclaurin series for $e^x + e^{3x}$ is

Solution: Adding the Maclaurin series for e^x and e^{3x} gives

$$e^x + e^{3x} = \sum_{j=0}^{\infty} \frac{x^j}{j!} + \sum_{j=0}^{\infty} \frac{(3x)^j}{j!} = \sum_{j=0}^{\infty} \frac{\left(1+3^j\right)}{j!} x^j .$$

(A) $\sum_{j=0}^{\infty} \frac{3^j}{j!} x^j$ [B) $\sum_{j=0}^{\infty} \frac{\left(1+3^j\right)}{j!} x^j$] (C) $\sum_{j=0}^{\infty} \frac{j!}{\left(1+3^j\right)} x^j$ (D) $\sum_{j=1}^{\infty} \frac{\left(1+3^j\right)}{j!} x^{j-1}$

(E) NOTA

20. Evaluating the improper integral $\lim_{b\to\infty} \int_2^b \frac{1}{x\ln(x)} dx$ shows that $\sum_{j=2}^{\infty} \frac{(-1)^j}{j\ln(j)}$

Solution: Since the improper integral does not exist, we have that the series is not absolutely convergent, but this is not one of the choices (A)-(D) so it is (E).

(A) sums to $\ln(\ln(2))$. (B) converges absolutely. (C) diverges

(D) converges conditionally. [(E) NOTA]

21. Applying the alternating series test to $\sum_{j=2}^{\infty} \frac{(-1)^j}{j\ln j}$ shows this series

Solution: The n^{th} of this series goes to 0 and it alternates in sign, so it is a convergent series.

(A) converges absolutely. [(B) converges.] (C) diverges.
(D) does not converge absolutely. (E) NOTA.

22. Evaluate $\sum_{n=0}^{\infty} \frac{2^n x^n}{n!}$.

Solution: (E) since we are given a power series in the variable x and (A) through (D) are constants.

(A) e (B) e^2 (C) $2e$ (D) e^{-2} [(E) NOTA]

23. Evaluate $\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n2^n}$.

Solution: This is the Taylor series for $\ln(1+x)$ evaluated at $x = -\frac{1}{2}$. Note that $\ln\left(\frac{1}{2}\right) = -\ln(2)$ and $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$, so letting $x = -\frac{1}{2}$ gives $-\ln(2) = \sum_{n=1} \frac{(-1)^{n+1}}{n} \cdot \left(-\frac{1}{2}\right)^n = \sum_{n=1} \frac{(-1)^{2n+1}}{n \cdot 2^n}$.

(A) $\ln(2)$ [(B) $-\ln(2)$] (C) 0 (D) $\ln(3)$ (E) NOTA

24. Which of the following describes the convergence of $\sum_{k=0}^{\infty} \frac{(-2)^k}{k^3}$.

Solution: By the ratio test this series diverges. Consider,

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{\frac{(-2)^{k+1}}{(k+1)^3}}{\frac{(-2)^k}{k^3}}\right| = 2\left(\frac{k}{k+1}\right)^3 \to 2 \text{ as } k \to \infty.$$ Since this limit is greater than 1 then by the ratio test the series is divergent.

(A) converges (B) converges absolutely (C) converges conditionally

[(D) diverges] (E) NOTA

25. Evaluate $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}$.

Solution: The Maclaurin expansion for $\sin(x)$ is given by $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$, thus the sum of the series is $\sin(1)$.

(A) 0 [(B) $\sin(1)$] (C) $\cos(1)$ (D) diverges (E) NOTA

26. The interval of convergence of $\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{x}{3}\right)^k$ is

Solution: For $a_k = \frac{1}{k}\left(\frac{x}{3}\right)^k$ consider the ratio

$\left|\frac{a_{k+1}}{a_k}\right| = \frac{1}{1+\frac{1}{k}}\left|\frac{x}{3}\right| \to \left|\frac{x}{3}\right|$ as $k \to \infty$. By the ratio test this series converges when

$\left|\frac{x}{3}\right| < 1$, or when $|x| < 3$. Now we only have to check the end points. At $x = 3$ this series reduces to the harmonic series which is divergent. At $x = -3$ it reduces to the alternating series $\sum_{k=1}^{\infty}\frac{(-1)^k}{k}$ which is convergent by the alternating series test. Thus, [-3,3) is the interval of convergence.

[(A) $[-3,3)$] (B) $\left[-\frac{1}{3},\frac{1}{3}\right)$ (C) $(-3,3]$ (D) $[-1,1)$ (E) NOTA

27. What are the terms up to degree 4 in the Maclaurin series of $\frac{\sin(x)}{1-x}$?

Solution: Either divide $\frac{\sin(x)}{1-x} = \frac{x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots}{1-x}$ which give up to 4^{th} order terms $x+x^2+\frac{5}{6}x^3+\frac{5}{6}x^4$ or multiple the Maclaurin expansions of $\frac{1}{1-x}$ and $\sin(x)$, $\left(1+x+x^2+x^3+\cdots\right)\left(x-\frac{x^3}{6}+\frac{x^5}{120}-\cdots\right)$ which in either case gives $\frac{\sin(x)}{1-x}$ as $x+x^2+\frac{5}{6}x^3+\frac{5}{6}x^4$ up to terms of order four.

(A) $x-x^2-\frac{7}{6}x^3-\frac{5}{6}x^4$ (B) $x-x^2-\frac{x^3}{6}+\frac{x^4}{6}$ [(C) $x+x^2+\frac{5}{6}x^3+\frac{5}{6}x^4$]

(D) $x+x^2+\frac{x^3}{6}+\frac{x^4}{6}$ (E) NOTA

28. At the first of each month, for ten years, \$1,000 is deposited into a saving account earning 6% a year compounded monthly. How much money is in this account when the last deposit is made? Round to nearest dollar.

Solution: Here S_n is the amount in the account at the start of each month.

$$s_1 = 1000.$$

$$s_2 = 1000\left(1 + \frac{0.06}{12}\right) + 1000.$$

$$s_3 = [1000(1.005) + 1000]1.005 + 1000$$

$$= 1000\left[(1.005)^2 + 1.005 + 1\right].$$

$$\vdots$$

$$s_{120} = \sum_{k=0}^{120} 1000(1.005)^{k-1}$$

$$= \frac{1000\left((1.005)^{120} - 1\right)}{1.005 - 1}$$

$$\approx 163{,}879.$$

(A) \$6,958,240 (B) \$165,699 (C) \$48,000 [(D) \$163,879] (E) NOTA

29. Which is true of the series $\sum_{k=1}^{\infty} \frac{1}{k^{3/2}}$?

Solution: This is a p-series with p=3/2 > 1 and hence a convergent series.

[(A) it converges.] (B) its sum is $\frac{3}{\pi}$. (C) it is conditionally convergent

(D) it is divergent. (E) NOTA

30. Which is true for the sequence $\left\{\frac{5}{3 + (-1)^n}\right\}$?

Solution: This sequence oscillates between the values 5/4 and 5/2 and hence it is divergent by oscillation.

(A) it converges to $\frac{5}{3}$. (B) it is unbounded. [(C) it is divergent by oscillation]

(D) it converges to $\frac{15}{4}$ (E) NOTA

TIEBREAKER:

Solution: First note that the ν^{th} term of this series is given by

$$\begin{aligned} a_\nu &= a^{\ln(\nu)} \\ &= e^{\ln(a)\ln(\nu)} \\ &= e^{\ln(\nu)\ln(a)} \\ &= \nu^{\ln(a)}. \end{aligned}$$

Thus, this is a $p-$series with $p = \ln(a)$ and it converges when $p = \ln(a) < -1$. This is true for all $a < e^{-1}$.