Vector Calculus

1. The vector $\overline{V}$ has a length of 4 and makes an angle of $\pi/6$ with the positive x axis. Write $\overline{V}$ as a linear combination of the unit vectors $\overline{i}$ and $\overline{j}$.

A. $\overline{V} = \frac{4\pi}{3}\overline{i} + \frac{\pi}{2}\overline{j}$

B. $\overline{V} = -2\sqrt{3}\,\overline{i} - \overline{j}$

C. $\overline{V} = 2\overline{i} - 2\sqrt{3}\,\overline{j}$

D. $\overline{V} = 2\sqrt{3}\,\overline{i} + 2\overline{j}$

E. NOTA

Solution: D

$$\overline{V} = 4\cos\pi/6\,\overline{i} + 4\sin\pi/6\,\overline{j}$$

$$= 4\,\frac{\sqrt{3}}{2}\,\overline{i} + 4\cdot\tfrac{1}{2}\,\overline{j}$$

$$= 2\sqrt{3}\,\overline{i} + 2\overline{j}$$

Vector Calculus

2. Find the magnitude of $\bar{V}$.

$$\bar{V} = 2\bar{i} - 3\bar{j} + 6\bar{k}$$

A. 5

B. 7

C. 9

D. $\sqrt{31}$

E. NOTA

Solution: B

$$\|\bar{V}\| = \sqrt{4+9+36} = \sqrt{49} = 7$$

Vector Calculus

3. find a unit vector normal to $f(x)$ at $(1,1)$.

$$f(x) = x^3 + 3$$

A. $\pm \frac{1}{\sqrt{10}}(3\bar{i} - \bar{j})$

B. $\pm \frac{1}{\sqrt{10}}(3i + j)$

C. $\pm \frac{1}{\sqrt{10}}(i - 3j)$

D. $\pm \frac{1}{\sqrt{10}}(i + 3j)$

E. NOTA

Solution A

$$f'(x) = 3x^2$$

$$f'(1) = 3 \qquad \bar{V}_{TAN} = i + 3j$$

$$\bar{A} \cdot \bar{B} = \|A\| \cdot \|B\| \cos\theta$$

$$V_{TAN} \cdot \bar{V}_\perp = 0$$

$$V_\perp = ai + bj \qquad V_{TAN} = i + 3j$$

$$a + 3b = 0$$

$$a = -3b$$

$$b = 1 \quad a = -3$$

$$\bar{V}_\perp = -3i + j$$

$$U_{V_\perp} = \frac{-3}{\sqrt{10}} i + \frac{1}{\sqrt{10}} j$$

Vector Calculus

4. Find the direction cosines of the vector $\bar{V}$.

$$V = 3\bar{i} + 4\bar{j} + 5\bar{k}$$

A. $3/5$, $4/5$, 1

B. $9/5\sqrt{2}$, $16/5\sqrt{2}$, $25/\sqrt{2}$

C. $3/5\sqrt{2}$, $4/5\sqrt{2}$, $1/\sqrt{2}$

D. $1/\sqrt{5}$, $3/\sqrt{5}$, $4/\sqrt{5}$

E. NOTA

Solution: C

$$\|V\| = \sqrt{9 + 16 + 25} = \sqrt{50} = 5\sqrt{2}$$

$$U_v = \frac{3}{5\sqrt{2}} i + \frac{4}{5\sqrt{2}} j + \frac{5}{5\sqrt{2}} k$$

cosines

Vector Calculus

5 A particle traveled from $A=(6,2,9)$ to $B=(126, 26, 153)$ in 24 seconds with a constant velocity. Find its position after 3 seconds from the start.

A. [21, 5, 27]
B. [7, 5/3, 9]
C. [18, 4, 25]
D. [28, 12, 34]
E. NOTA

Solution: A

$$S(t) = S(A) + V(t-a)$$
$$(126, 26, 153) = (6,2,9) + V(24)$$
$$(120, 24, 144) = V(24)$$
$$(5,1,6) = V$$

$$S(t) = (6,2,9) + t(V)$$
$$S(3) = (6,2,9) + 3(5,1,6)$$
$$S(3) = [21, 5, 27]$$

Vector
Calculus

6 The length of $\vec{u} = 4$ and the length of $\vec{v} = 9$. The angle between $\vec{u}$ and $\vec{v}$ is $60°$. Find the length of $\vec{u} + \vec{v}$.

A. $\sqrt{87}$

B. $\sqrt{113}$

C. $\sqrt{133}$

D. $\sqrt{155}$

E. NOTA

Solution: $\sqrt{133}$ C

$W = u + v$

$|W| = \sqrt{W \cdot W}$

$|\vec{W}|^2 = (u+v) \cdot (u+v)$

$|\vec{W}|^2 = |\vec{u}|^2 + 2\vec{u} \cdot \vec{v} + |\vec{v}|^2$

$|\vec{W}|^2 = 16 + 2|u||v| \cos\theta + 81$

$2|4||9| \cos 60$

$16 + 36 + 81$

$|\vec{W}|^2 = 133$

$\vec{w} = \sqrt{133}$

Vector Calculus

7

Vector $\vec{V} = [72, 54]$ is rotated counter clockwise 30°. Find the rotated vector.

A. $[36 - 27\sqrt{3}, -36\sqrt{3}, 27]$

B. $[36\sqrt{3} - 27, 36 - 27\sqrt{3}]$

C. $[-36\sqrt{3} + 27, -36 - 27\sqrt{3}]$

D. $[-36\sqrt{3} - 27, -36 + 27\sqrt{3}]$

E. NOTA

Solution: D

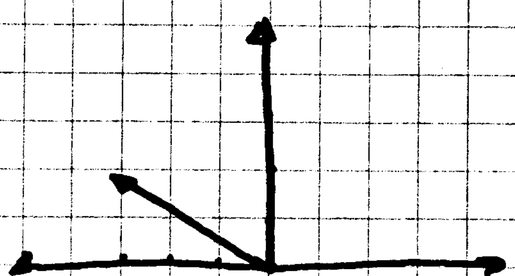
$R(\vec{v}) = R(xi + yj) = xRi + yRj$

$Ri = [\cos\theta, \sin\theta] \qquad Rj = [-\sin\theta, \cos\theta]$

$R(v) = -72(\cos 30, \sin 30] + 54(-\sin 30, \cos 30]$

$[-36\sqrt{3}, -36] + [-27, \frac{27\sqrt{3}}{2}]$

$= [-36\sqrt{3} - 27, -36 + 27\sqrt{3}]$



Vector
CALCULUS

8

Vector $\vec{U} = [4, 1]$ is rotated counter clockwise until it points in the direction of $\vec{V} = [-3, 8]$. Find the angle of rotation θ.

A. $\theta = \cos^{-1}\left(\frac{-4}{\sqrt{225}}\right)$

B. $\theta = 92°$

C. $\theta = \cos^{-1}\left(\frac{-4}{\sqrt{17}+\sqrt{73}}\right)$

D. $\theta = \cos^{-1}\left(\frac{-4}{\sqrt{1241}}\right)$

E. NOTA

Solution: D

$$\vec{U} \cdot \vec{V} = |\vec{U}||\vec{V}| \cos\theta$$

$$\cos\theta = \frac{\vec{U} \cdot \vec{V}}{|\vec{U}||\vec{V}|} = \frac{-4}{\sqrt{17}\sqrt{73}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{17}\sqrt{73}}\right)$$

$$\begin{array}{r} 73 \\ 17 \\ \hline 511 \\ 73 \\ \hline 1241 \end{array}$$

Vector Calculus

9. Find the projection of $\vec{A}$ onto $\vec{B}$ and the component of $\vec{A}$ orthogonal to $\vec{B}$.

$$\vec{A} = 3i + 4j - 2k$$

$$\vec{B} = 6i - 5j + 3k$$

A. $\langle 129/35, 24/7, -58/12 \rangle, \langle -24/35, 4/7, -12/35 \rangle$

B. $\langle 3/5, 4/5, -2/5 \rangle, \langle 6/5, -1, 3/5 \rangle$

C. $\langle 129/35, 24/7, -58/12 \rangle, \langle 9, -1, 1 \rangle$

D. $\langle -24/35, 4/7, -12/35 \rangle, \langle 129/35, 24/7, -58/35 \rangle$

E. NOTA

Solution: D

$$\text{Proj}\, A_B = \frac{A \cdot B}{\|B\|^2} B$$

$$\left(\frac{18-20-6}{36+25+9}\right) 6i - 5j + 3k$$

$$-\frac{4}{35} \qquad \frac{-8}{70}(6i - 5j + 3k)$$

$$\underbrace{(3i + 4j - 2k) - \left[\frac{-24}{35} i + \frac{20}{35} j - \frac{12}{35} k\right]}_{\text{}}$$

orthogonal component

projection

VECTOR CALCULUS

10 If $(\vec{u} \times \vec{v}) = [3, -8, 3]$, find $[-1\vec{u} + 3\vec{v}] \times [\vec{u} + 3\vec{v}]$.

A. [6, -16, 6]

B. [-18, 48, -18]

C. CAN NOT WORK WITH Information Provided

D. [-36, 96, -36]

E. NOTA

Solution: B

$$(-\vec{u} + 3\vec{v}) \times (\vec{u} + 3\vec{v}) =$$

$$\cancel{-\vec{u} \times \vec{u}}^{\,0} - 3\,\overline{u} \times \overline{v} + \underbrace{3\,\overline{v} \times \overline{u}}_{-3\,u \times v} + \cancel{9\,\overline{v} \times \overline{v}}^{\,0}$$

$$-6(\overline{u} \times \overline{v}) = -6(3, -8, 3) =$$

$$[-18, 48 -18]$$

11 Find the area of a triangle with vertices at $P(-9,7,2)$, $Q(6,-5,7)$, $R(3,1,6)$.

A. $\sqrt{3240}$

B. $\sqrt{3240}/2$

C. $\sqrt{3240}/4$

D. $\sqrt{2916}/2$

E. NOTA

Solution: B

$\cancel{\overrightarrow{QR} \neq \overrightarrow{PQ}}$

$\bar{u} = \overrightarrow{QP} = -15i + 12j - 5k$

$\bar{v} = \overrightarrow{QR} = -3i + 6j - k$

$\text{Area } \Delta = \frac{1}{2}|u \times v|$

$$\bar{u} \times \bar{v} \begin{vmatrix} i & j & k \\ -15 & 12 & -5 \\ -3 & 6 & -1 \end{vmatrix} = 18i + 0j - 54k$$

$$|u \times v| = \sqrt{324 + 2916} = \frac{\sqrt{3240}}{2}$$

Vector Calculus

12 Find the volume of a parallelpiped having $\vec{A} = 3\bar{i} - 5\bar{j} + \bar{k}$ $\vec{B} = 2\bar{j} - 2\bar{k}$ and $\vec{C} = 3\bar{i} + \bar{j} + \bar{k}$ as adjacent edges.

A. 20

B. 24

C. 30

D. 36

E. NOTA

Solution 36 (D)

$$V = |\bar{A} \cdot (\bar{B} \times \bar{C})|$$

$$\begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} = 6 + 30 - 6 + 6$$
$$= 36$$

Vector
Calculus

13. Given $\bar{s}(t) = {}^1\!/\!_t\,\bar{i} - \bar{j} + (\ln t)\bar{k}$ and
$\bar{u}(t) = t^2\bar{i} - 2t\,\bar{j} + \bar{k}$

Find $D_t\,[\bar{s}(t) \cdot \bar{u}(t)]$

A. $t^2 i - 2t j + 3k$

B. $3 + 1/t$

C. $t^2 + 2t + 2$

D. $1 + \frac{1}{2t^2}$

E. NOTA

Solution: B

$$D_t(\bar{s}(t) \cdot \bar{u}(t)) = \bar{s}(t) \cdot \bar{u}'(t) + \bar{u}(t) \cdot \bar{s}'(t)$$

$$\left(\tfrac{1}{t}\,i - j + (\ln t)k\right) \cdot (2t\,i - 2j + 0k) +$$
$$(t^2 i - 2t\,j + k) \cdot \left(-\tfrac{1}{t^2} + 0j + \tfrac{1}{t}k\right)$$

$$2 + 2 - 1 + 1/t = \underline{\underline{3 + 1/t}}$$

Vector
CALCULUS

14 Given $\overline{V}(t) = t^2\bar{i} - 2t\bar{j} + k$

find $D_t[\overline{V}(t) \times \overline{V}'(t)]$

A. $2\bar{j} + 4t\bar{k}$

B. $2\bar{i} - 2\bar{j} - 4t\bar{k}$

C. $2t\bar{i} - 2t\bar{j} + 4t\bar{k}$

D. $8t^2 i - 4tj + 5t\bar{k}$

E. NOTA

Solution A

$\overline{V}(t) = t^2 i - 2tj + k$

$V'(t) = 2ti - 2j$

$$\overline{V}(t) \times V'(t) = \begin{vmatrix} i & j & k \\ t^2 & -2t & 1 \\ 2t & -2 & 0 \end{vmatrix}$$

$= 2tj - 2t^2k + 4t^2k + 2i + 0j$

$D_t = 2j - 4tk + 8tk$

$= 2\bar{j} + 4t\bar{k}$

Vector
CALCULUS

15

Given $\overline{r}'(t) = 4\cos 2t\,\overline{i} - 4\cos t\,\overline{j} + \frac{1}{1+t^2}\overline{k}$

$\overline{r}(0) = 2\overline{i} - 4\overline{j} + \overline{k}$

find $\overline{r}(t)$

A. $\overline{r}(t) = \left(\frac{\cos 4t}{2} + 2\right)\overline{i} - (\sin 4t + 4)\overline{j} + (\arccos(t) + 1)\overline{k}$

B. $\overline{r}(t) = \left(\frac{\sin 8t}{2} + 2\right)\overline{i} - (\sin 4t + 4)\overline{j} + (\text{arc}\sec(t) + 1)\overline{k}$

C. $\overline{r}(t) = (2\sin 2t + 2)\overline{i} - (4\sin t + 4)\overline{j} + (\arctan(t) + 1)\overline{k}$

D. $\overline{r}(t) = (2\sin 2t + 2)\overline{i} - 4(\sin t - 1)\overline{j} + (\text{arc}\cot t + 1)\overline{k}$

E. NOTA

Solution: C

$\int r'(t) = 2\sin 2t \,\overline{i} - (4\sin t + c_2)\overline{j} + (\arctan t + c_3)k$

$r(0) = 2\overline{i} - 4\overline{j} + \overline{k}$

$2 = 2\sin 2(0) + c_1$ $\qquad -4 = (4\sin t + c_2)$

$c_1 = 2$ $\qquad -4 = -c_2$

$1 = \arctan 0 + c_3$

$1 = c_3$

$\therefore$

$r(t) = (2\sin 2t + 2)\overline{i} - (4\sin t + 4)\overline{j} + (\arctan t + 1)k$

Vector
Calculus

16 A Particle moves along a Plane curve described by

$$P(t) = 4 \sin \frac{t}{4} \bar{i} + 4 \cos t/4 \, j$$

find the Speed of the particle @ any time t.

A. $\cos t/4 \, \bar{i} - \sin t/4 \, \bar{j}$

B. $1/4 \sin t/4 \, \bar{i} - 1/4 \cos t/4 \, \bar{j}$

C. $1/4 \cos t/4 \, \bar{i} - 1/4 \sin t/4 \, \bar{j}$

D. 1

E. NOTA

Solution D

$$P'(t) = \text{velocity} = 1/4 \, 4 \cos t/4 \, \bar{i} - 1/4 \, 4 \sin t/4 \, \bar{j}$$

Speed = $\|P'(t)\|$ = Magnitude of Velocity

Speed $= \cos^2 t/4 + \sin^2 t/4 = 1$

Vector Calculus

17

11. Find the unit tangent vector $\vec{T}$ to the curve
$y^3x^2 + 2x + 3y + xy = -3$ at Point $P(1,-1)$
pointing up. (Pointing up means y component of $\vec{T}$ is positive)

A $1/\sqrt{50}\,[-7, 1]$

B $1/\sqrt{50}\,[7, 1]$

C $1/\sqrt{50}\,[-1, +7]$

D $1/\sqrt{50}\,[1, 7]$

E NOTA

Solution: B

$$f(x,y) = y^3x^2 + 2x + 3y + xy = -3$$

$$n = \nabla f = [2xy^3 + 2 + y,\ 3y^2x^2 + 3 + x]$$

@ $(1,-1)$

$$n = [-1, 7]$$

Rotation of $N = T$

$$T = [-7, -1]$$

With Positive y Component

$[7, 1]$ Tangent

$\frac{1}{\sqrt{50}}[7, 1]$ Unit Tangent

W2
#6

18. Find the line integral of

$(2x+2y+1)\,dx + (2x+4)\,dy$ along the curve C given by $x(t)=t$, $y(t)=t^2$, $0 \le t \le 2$.

A. 38

B. 36

C. 21

D. 10

E. NOTA

Solution = A

The line integral is

$\int (f\,dx + g\,dy)$

$x(t) = t \qquad y(t) = t^2$

$dx = dt \qquad dy = 2t\,dt$

$\int_0^2 (2t+2t^2+1)\,dt + (2t+4)\,2t\,dt$

$t^2 + \frac{2}{3}t^3 + t + \frac{4}{3}t^3 + 4t^2 \Big|_0^2$

$4 + \frac{16}{3} + 2 + \frac{32}{3} + 16$

$4 + 16 + 2 + 16 = 38$

Vector Calculus

19. Let $\Phi(x, y)$ be the potential of the vector field $F(x,y) = [3y-4, 3x+2]$. So $\Phi(0,0) = 2$. Find $\Phi(1,2)$.

A. 2

B. 4

C. 6

D. 8

E. NOTA

Solution: D

$$F(x,y) = [3y-4, 3x+2] = \nabla \Phi$$

$$\Phi_x = 3y-4 \Rightarrow \Phi = 3xy - 4x + C_y$$

$$\Phi_y = 3x+2 \Rightarrow$$

$$3x+2 = \Phi_y = 3x + C'(y)$$

$$2 = C'_y$$

$$\Phi_y = 3x+2 \Rightarrow \Phi = 3xy + 2y + C(x)$$

$$\Phi = 3xy - 4x + 2y + 2$$

$$\Phi(1,2) = 3 \cdot 1 \cdot 2 - 4 \cdot 1 + 2 \cdot 2 + 2$$

$$= 6 - 4 + 4 + 2$$

$$= 8$$

Vector Calculus

20. Which of the following vector fields is/are conservative?

A. $F(x,y) = 2xy\,\bar{i} + x^2\,\bar{j}$

B. $F(x,y) = xe^{x^2y}(2y\,\bar{i} + x\,\bar{j})$

C. $F(x,y) = \dfrac{x\bar{i} + y\,\bar{j}}{x^2+y^2}$

D. All are conservative

E. NOTA

Solution D

$F(x,y) = \bar{M}i + \bar{n}j$

$$\frac{\partial M}{\partial y} \cdot \frac{\partial n}{\partial x}$$

(A) $2x = 2x$

conservative

(B) $2yxe^{x^2y}\,\bar{i} + x^2e^{x^2y}\,\bar{j}$

$$2x^3y\,e^{x^2y} + 2xe^{x^2y} = 2x^3ye^{x^2y} + 2xe^{x^2y}$$

conservative

(C) $$\frac{-x(2y)}{(x^2+y^2)^2} \doteq -\frac{y\,2x}{(x^2+y^2)^2}$$

conservative

21. Find the curl of the vector field F at the indicated point.

$$F(x,y,z) = e^x \sin y\, \bar{i} - e^x \cos y\, \bar{j}$$

$$P = (0, 0, 3)$$

A. $2\bar{i} - 2\bar{j}$

B. $-2\bar{k}$

C. $2\bar{i} - 2\bar{j} - 2\bar{k}$

D. $2j - 2k$

E. NOTA

Solution: B

$$\text{Curl} = \nabla \times \bar{F}(x,y,z)$$

$$\text{Curl} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y & -e^x \cos y & 0 \end{vmatrix} =$$

$$0i + \frac{\partial}{\partial z} e^x \sin y\, j + \frac{\partial}{\partial x}(-e^x \cos y) k$$

$$- \left[\frac{\partial}{\partial y} e^x \sin y\, k - \frac{\partial}{\partial z} e^x \cos y\, i + 0 j \right]$$

$$(-e^x \cos y - e^x \cos y)\, k \qquad @\, (0,0,3)$$

$$-2\bar{k}$$

22. Find the conservative vector field for the potential function $f(x,y,z) = \frac{y}{z} + \frac{z}{x} - \frac{xz}{y}$.

A. $F(x,y,z) = \left(-\frac{z}{x^2} - \frac{z}{y}\right)\overline{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\overline{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\overline{k}$

B. $F(x,y,z) = \left(-z/x^2\right)\overline{i} + \left(xz/y^2\right)\overline{j} - y/z^2\,\overline{k}$

C. $F(x,y,z) = \left(-\frac{z}{x^2} - \frac{z}{y} + \frac{y}{z}\right)\overline{i} + \left(\frac{xz}{y^2} + \frac{1}{z^2} - xz\right)\overline{j} + \left(-y/z^2 + 1/x^2 - \frac{x}{y}\right)\overline{k}$

D. $F(x,y,z) = \left(-z/x^2 + z/y\right)\overline{i} + \left(-1/z - \frac{xz}{y^2}\right)\overline{j} + \left(y/z^2 + \frac{1}{z}\right)\overline{k}$

E. NOTA

Solution: A

$$\nabla f(x,y,z) = f_x(x,y,z)\,\overline{i} + f_y(x,y,z)\,\overline{j} + f_z(x,y,z)\,\overline{k}$$
$$= \left(-\frac{z}{x^2} - \frac{z}{y}\right)\overline{i} + \left(\frac{1}{z} + \frac{xz}{y^2}\right)\overline{j} + \left(-\frac{y}{z^2} + \frac{1}{x} - \frac{x}{y}\right)\overline{k}$$

29. Find the work done by the vector field $F(x,y) = -y\bar{i} + x\bar{j}$ on the particle moving counterclockwise around the ellipse $5x^2 + 4y^2 = 2$.

A. 0

B. $2\sqrt{5}\pi/5$

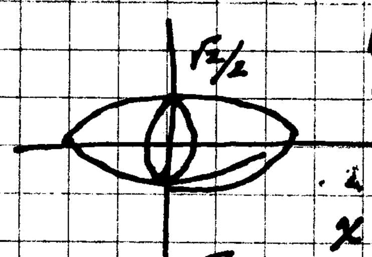
C. $5\sqrt{2}\pi/2$

D. $2\pi/5$

E. NOTA

Solution B

$$\frac{x^2}{2/5} + \frac{y^2}{1/2} = 1$$



$b^2 = 1/2 \quad b = 1/\sqrt{2} = \sqrt{2}/2 =$

$a^2 = 2/5 \quad a = \sqrt{2/5} = \frac{\sqrt{10}}{5} =$

$x = \sqrt{2/5}\cos t \quad \frac{dx}{dt} = -\frac{\sqrt{2}}{\sqrt{5}}\sin t$

$y = \sqrt{2}/2 \sin t \quad \frac{dy}{dt} = \frac{\sqrt{2}}{2}\cos t$

$$F(t) = \left[-\sqrt{2}/2 \sin t, \sqrt{2}/\sqrt{5}\cos t\right]$$

$$W = \int F \cdot V\,dt = \int_0^{2\pi} \left[-\frac{\sqrt{2}}{2}\sin t, \frac{\sqrt{2}}{\sqrt{5}}\cos t\right] \cdot \left[-\frac{\sqrt{2}}{\sqrt{5}}\sin t, \frac{\sqrt{2}}{2}\cos t\right] dt$$

$$= \int_0^{2\pi} \left(\frac{1}{\sqrt{5}}\sin^2 t + \frac{1}{\sqrt{5}}\cos^2 t\right) dt$$

$$= \int_0^{2\pi} 1/\sqrt{5}\,dt = \frac{t}{\sqrt{5}}\Big|_0^{2\pi} = \frac{2\pi}{\sqrt{5}}$$

Vector Calculus

24. Find the flux of F over the closed Surface. (Let N be the outward unit normal vector of the surface.)

$$F(x,y,z) = (x+y)\bar{i} + y\,\bar{j} + z\bar{k}$$

Surface, S: $z = 1 - x^2 - y^2$, $z = 0$

A. $3\pi/4$

B. $\pi/2$

C. $3\pi/2$

D. $3\pi/5$

E. NOTA

Solution: C

$$F(x,y,z) = (x+y)\bar{i} + y\bar{j} + zk$$

$$G(x,y,z) = 1 - x^2 - y^2 - z$$

$$g_x(x,y) = -2x \qquad g_y(x,y) = -2y$$

The rate of mass flow through the Surface

$$\iint F \cdot N\,dS = \iint_R F \cdot [-g_x(x,y)i - g_y(x,y)j) + k]\,dA$$

$$= \iint_R ((x+y)\bar{i} + y\bar{j} + zk) \cdot [2xi + 2yj + k]\,dA$$

$$\iint_R 2x^2 + 2xy + 2y^2 + z\,dA = \iint 2x^2 + 2xy + 2y^2 + 1 - x^2 - y^2\,dA$$

$$\iint \underbrace{x^2 + 2xy + y^2}_{r^2} + 1\; r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 [r^2 + 2r^2 \sin\theta\cos\theta + 1]\,r\,dr\,d\theta$$

$$\int_0^{2\pi} \frac{r^4}{4} + \frac{2r^4}{4}\sin\theta\cos\theta + \frac{r^2}{2}\Big|_0^1 d\theta = \int_0^{2\pi} \frac{3}{4} + \frac{1}{2}\sin\theta\cos\theta\,d\theta = 3\pi/2$$

25

Let $F(x,y) = [7x^3+9xy,\ 8x+8y^3]$. find $\int_C F\cdot n\,ds$

where C is a square with vertices at
$(1,-1), (1,1), (-1,1), (-1,-1)$.

A. 60

B. 50

C. 30

D. 20

E. NOTA

Solution: A

$$\int F\cdot n\,ds = \iint (\nabla\cdot F)\,dx\,dy$$

$$\int_{-1}^{1}\int_{-1}^{1}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\cdot\left[7x^3+9xy,\ 8x+8y^3\right]dx\,dy$$

$$\int_{-1}^{1}\int_{-1}^{1} 21x^2+9y+24y^2\,dx\,dy$$

$$\int_{-1}^{1} 7x^3+9xy+24xy^2\Big|_{-1}^{1}\,dy$$

$$7+9y+24y^2-[-7-9y-24y^2]$$

$$\int_{-1}^{1}(14+18y+48y^2)\,dy$$

$$14y+9y^2+16y^3\Big|_{-1}^{1}$$

$$14+9+16-[-14+9-16] = \underline{60}$$

Vector Calculus

26. Let $F(x,y) = [4+9(x-4)+5(y-8),\ 6+3(x-4)+(y-8)]$

Find $\int_C F \cdot n\,ds$, where C is the circle of radius 7 at (4,8).

A. 50π

B. 226π

C. 490π

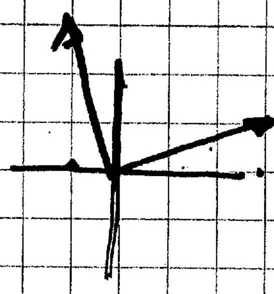
D. 500π

E. NOTA

Solution: C

Rotated Velocity Vector

$$\int_C F \cdot n\,ds = \int F \cdot R(V)\,dt$$



Parameterization of circle

$x-4 = 7\cos t \qquad \frac{dx}{dt} = -7\sin t$

$y-8 = 7\sin t \qquad \frac{dy}{dt} = 7\cos t$

$V = [-7\sin t,\ 7\cos t]$

$R(V) = [7\cos t,\ 7\sin t]$

$$\int_0^{2\pi} [4+9\cdot 7\cos t + 5(7\sin t),\ 6+3(7\cos t)+7\sin t] \cdot [7\cos t,\ 7\sin t]\,dt$$

with exception of $\cos^2 + \sin^2$ all $\int$ are 0

$441\pi + 49\pi = \underline{490\pi}$

Vector Calculus

27. find the outward flux of F through the surface of the solid bounded by the graphs of the equations.

$$F(x,y,z) = x\bar{i} + y\bar{j} + z\bar{k}$$

Surface S: $x^2+y^2+z^2=4$

A. 4π

B. 8π

C. 16π

D. 32π

E. NOTA

Solution

Flux across $S = \iint_S F \cdot N \, ds = \iiint \nabla F \, dV$

$$= \int_0^2 \int_0^{\pi} \int_0^{2\pi} 3\rho^2 \sin\phi \, d\theta \, d\phi \, d\rho = \int_0^2 \int_0^{\pi} 6\pi \rho^2 \sin\phi \, d\phi \, d\rho$$

$$\int_0^2 -6\pi \rho^2 \cos\phi \Big|_0^{\pi} d\rho$$

$$[-1-1]$$

$$\int_0^2 12\pi \rho^2 d\rho = \frac{12\pi \rho^3}{3} \Big|_0^2 = 32\pi$$

28. Given the field $F(x,y,z) = z^2\bar{i} + x^2\bar{j} + y^2 k$
and Surface, $S: \ z = 4 - x^2 - y^2, \ z \geq 0$
evaluate $\int_C F \cdot dr$.

A. 8π

B. 6π

C. 0

D. -2π

E. NOTA

Solution: C Stokes' Theorem

$$\int_C F \cdot dr = \iint_S (\text{curl} F) \cdot N \, ds$$

$$\text{curl of } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & x^2 & y^2 \end{vmatrix} = 2y\bar{i} + 2z\bar{j} + 2x\bar{k}$$

$$G(x,y,z) = x^2 + y^2 + z - 4$$

$$\nabla G(x,y,z) = 2x\,i + 2y\,j + k$$

$$\iint_S (\text{curl} F) \cdot N\,ds = \iint_R [2yi + 2zj + 2xk] \cdot [2xi + 2yj + k]\, dA$$

$$\iint 4xy + 4zy + 2x = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 4xy + 4y(4 - x^2 - y^2) + 2x \; dy\,dx$$

$$= \int_{-2}^{2} 4x\sqrt{4-x^2}\,dx = 0$$

Vector calculus

29. Given the position function

$$r(t) = a\cos\omega t\, i + b\sin\omega t\, j$$

find the normal component of acceleration at $t=0$

A. $ab\omega^2$

B. $ab\omega^3$

C. $b\omega^2$

D. $a\omega^2$

E. NOTA

Solution: D

$$r'(t) = -a\omega\sin\omega t\, i + b\omega\cos\omega t\, j$$

$$r''(t) = -a\omega^2\cos\omega t\, i - b\omega^2\sin\omega t\, j$$

$$a_N = \frac{|v \times a|}{|v|}$$

$$\begin{array}{ccc} i & j & k \\ -a\omega\sin\omega t & b\omega\cos\omega t & 0 \\ -a\omega^2\cos\omega t & -b\omega^2\sin\omega t & 0 \end{array}$$

$$\frac{-a\omega\sin\omega t - b\omega^2\sin\omega t\, k + a\omega^3\cos^2\omega t\, k}{(a^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t)^{1/2}}$$

$$\frac{a\omega^3 \cancel{b}}{\cancel{b}\omega} = \frac{a\omega^2}{\cancel{b}} = a\omega^2$$

30. find the equation of a tangent vector to the helix given by

$$r(t) = 2\cos t\,\bar{i} + 2\sin t\,\bar{j} + tk$$

A. $-2\sin t\,\bar{i} + 2\cos t\,\vec{j} + k$

B. $2\sin t\,\bar{i} + 2\cos t\,\bar{j}$

C. $2\sin t\,\bar{i} + 2\cos t\,\bar{j} + k$

D. $-2\sin t\,\bar{i} + 2\cos t\,j - k$

E. NOTA

Solution (A)

$$r'(t) = -2\sin t\,\bar{i} + 2\cos t\,\bar{j} + k$$

Vector Calculus

31. evaluate the definite integral

$$\int_0^{\pi/2} [a\cos t\,\bar{i} + a\sin t\,\bar{j} + \bar{k}]\,dt$$

A $\bar{a}\bar{i} + a\bar{j} + \frac{\pi}{2}\bar{k}$

B $-ai - aj + \frac{\pi}{2}k$

C $-a\bar{i} + a\bar{j} + \bar{k}$

D $ai - aj + \pi/2\,k$

E NOTA

Solution: A

$$\int_0^{\pi/2} [a\cos t\,\bar{i} + a\sin t\,j + k]\,dt$$

$$a\sin t\,i - a\cos t\,j + tk \Big|_0^{\pi/2}$$

$$ai + \pi/2\,k + aj$$

$$ai + aj + \pi/2\,k$$

Vector Calculus

32. find the directional derivative of $f(x,y) = 3x - 4xy + 5y$ at $(1,2)$ in the direction of $\vec{V} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$.

A. $(-5 + \sqrt{3})$

B. $\frac{1}{2}(-5 + \sqrt{3})$

C. $-\frac{5}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$

D. $\frac{\sqrt{3}}{2}\vec{i} - \frac{5}{2}\vec{j}$

E. NOTA

Solution: B

$$\nabla f = (3-4y)i + (-4x+5)j \overset{@1,2}{=} -5i + j$$

$$\frac{\nabla f \cdot V}{|V|} = \frac{(-5i+j)\cdot(\frac{1}{2}i + \frac{\sqrt{3}}{2}j)}{1}$$

$$-\frac{5}{2} + \frac{\sqrt{3}}{2} = \frac{-5+\sqrt{3}}{2}$$

Vector Calculus

33. Find the equation of the tangent plane to the hyperboloid given by

$$x^2+y^2-z^2=0 \quad @ \ (5,12,13)$$

A. $2x+2y-2z=8$

B. $-2x-2y-2z=8$

C. $5x+12y+13z=0$

D. $-5x+12y-13z=0$

E. NOTA

Solution E.

$$F(x,y,z) = x^2+y^2-z^2$$

$F_x = 2x \quad @ \ 5, 12, 13 = 10$

$F_y = 2y \quad @ \ 5, 12, 13 = 24$

$F_z = -2z \quad @ \ 5, 12, 13 = -26$

Direction numbers $(5, 12, -13)$

The required equation is then

$$5(x-5)+12(y-12)+(-13)(z-13)=0$$

$$5x-12+12y-144-13z+169=0$$

$$\cancel{2x+10+2y+24=2z+26}$$

$$\underline{5x+12y-13z=0}$$

$$2x+2y-2z=0$$