2003 Mu Alpha Theta National Convention
Matrices and Vectors Topic Test – Alpha Division

1. If $2\begin{pmatrix} 1 & -3 \\ x & 2 \end{pmatrix} - 4\begin{pmatrix} y & -2 \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 4 & -8 \end{pmatrix}$, what is the value of $x+y$?

(A) -1 (B) $-\frac{5}{4}$ (C) 0 (D) 1 (E) NOTA

2. Evaluate $|r-s|$ if $r[1,2,3]+s[1,-2,4]+t[-1,0,-2]=[-2,5,3]$.

(A) $\frac{5}{2}$ (B) 3 (C) $\frac{7}{2}$ (D) 2 (E) NOTA

3. What is the rank of $\begin{pmatrix} 2 & 4 & 0 & -2 \\ 0 & 1 & -1 & 0 \\ 1 & 2 & -1 & -1 \end{pmatrix}$?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

4. To the nearest tenth of a degree, what is the measure of the smallest positive angle the vector $[2, \sqrt{6}]$ makes with the positive x-axis?

(A) 54.7° (B) 35.3° (C) 39.2° (D) 50.8° (E) NOTA

5. Find the entry in the third row, fourth column of the product.

$$\begin{pmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 6 & 10 \\ 6 & 28 & 496 & 8128 \\ 126 & 153 & 688 & 1206 \\ 220 & 284 & 1184 & 1210 \end{pmatrix} \begin{pmatrix} 2 & 3 & 5 & 7 & 11 & 13 & 17 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 \\ 1 & 4 & 9 & 13 & 19 & 27 & 32 \\ 1 & 1 & 2 & 5 & 14 & 42 & 132 \end{pmatrix}$$

(A) 10611 (B) 18304 (C) 21002 (D) 47577 (E) NOTA

6. Given two nonzero vectors $\vec{a}$ and $\vec{b}$ in the xy-plane such that $\|\vec{a}+\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2$ it can be concluded that:

(A) $\vec{a}$ and $\vec{b}$ are orthogonal. (B) $\|\vec{a}\| > \|\vec{b}\|$

(C) $\|\vec{a}\| + \|\vec{b}\| < 1$ (D) $\frac{\|\vec{a}\|\|\vec{b}\|}{\vec{a}\cdot\vec{b}} < 0$ (E) NOTA

7. If A is a singular matrix, then $|A|$ equals:

(A) 1 (B) 0 (C) ± 1 (D) -1 (E) NOTA

8. What is the cosecant of the angle between $[3,4]$, and $[8,15]$?

(A) $\frac{24}{13}$ (B) $\frac{25}{13}$ (C) $\frac{84}{13}$ (D) $\frac{85}{13}$ (E) NOTA

9. What is the sum of the eigenvalues of $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -3 \\ 0 & 0 & 5 \end{pmatrix}$?

(A) -2 (B) 0 (C) 6 (D) 8 (E) NOTA

10. Let $\vec{x} = [1, 2, 3]$, $\vec{y} = [4, 5, 6]$, and $\vec{z} = [7, 8, 9]$. Find the value of

$$\frac{((\vec{x} \cdot \vec{y})\vec{z}) \cdot ((\vec{x} \cdot \vec{z})\vec{y})}{\|(\vec{y} \cdot \vec{z})\vec{x}\|^2}$$

(A) $\frac{135}{784}$ (B) $\frac{75}{61}$ (C) $\frac{400}{427}$ (D) Undefined. (E) NOTA

11. If $A = \begin{pmatrix} 4 & -4 \\ -3 & 8 \end{pmatrix}$, $|B| = 17$, and $ABC = \begin{pmatrix} 12 & -3 \\ 0 & 51 \end{pmatrix}$, then $|C| = m/n$, where m and n are relatively prime natural numbers. What is $m + n$?

(A) 14 (B) 47 (C) 158 (D) 632 (E) NOTA

12. Given that $\vec{x} \cdot \vec{y} = 7$ and $\|\vec{x} + \vec{y}\| = 8$, find the value of $\|\vec{x} - \vec{y}\|$.

(A) $2\sqrt{3}$ (B) 6 (C) $\sqrt{57}$ (D) $\sqrt{113}$ (E) NOTA

13. Evaluate: $\begin{pmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{pmatrix}^{2002}$

(A) $\begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}$ (B) $\begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$

(C) $\begin{pmatrix} \sqrt{3}/2 & 1/2 \\ -1/2 & \sqrt{3}/2 \end{pmatrix}$ (D) $\begin{pmatrix} -\sqrt{3}/2 & -1/2 \\ 1/2 & -\sqrt{3}/2 \end{pmatrix}$ (E) NOTA

14. Find $a - b$ if $[3, b, a]$ is parallel to $[6, 2b, 14]$ and orthogonal to $[17b, -2, a]$.

(A) 5 (B) 7 (C) 6 (D) 8 (E) NOTA

15. The set

$$\left\{ \begin{pmatrix} 4 & 6 \\ 6 & 0 \end{pmatrix}, \begin{pmatrix} 7 & 1 \\ 4 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 5 & 8 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ -1 & 9 \end{pmatrix} \right\}$$

has m symmetric matrices and n skew-symmetric matrices. Find $(m - n)^m$.

(A) -1 (B) 0 (C) 4 (D) 1 (E) NOTA

16. Which of the following is an eigenvector of $\begin{pmatrix} 8 & -21 \\ 3 & -8 \end{pmatrix}$?

(A) $[3, 7]$ (B) $[1, 1]$ (C) $[3, 1]$ (D) $[4, 2]$ (E) NOTA

17. What is the area of the triangle with vertices $(1, 6, -12)$, $(0, -7, 5)$, and $(2, 0, 4)$?

(A) $\frac{\sqrt{3918}}{2}$ (B) $\frac{99}{\sqrt{2}}$ (C) 41 (D) $\frac{\sqrt{12686}}{2}$ (E) NOTA

18. Let $\vec{u} = [2, 7, 6]$ and $\vec{v} = [-7, 0, -5]$. What is the vector projection of $\vec{u}$ along $\vec{v}$?

(A) $-\frac{1}{\sqrt{6586}}[-7, 0, -5]$ (B) $-\frac{22}{37}[-7, 0, -5]$

(C) $-\frac{18}{163}[-7, 0, -5]$ (D) $-\frac{44}{89}[-7, 0, -5]$ (E) NOTA

19. Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Find the trace of $\sum_{i=0}^{n} A^i$.

(A) $2^{n+1} + n$ (B) $2n + 1$ (C) 0 (D) $2^{n(n+1)/2}$ (E) NOTA

20. Which of the following planes is perpendicular to $5x - 7y + 3z = 8$?

(A) $7x - 5y + 3z = 4$ (B) $5x + 3z = 7y + 1$

(C) $z = (4x - y + 3)/5$ (D) $6z = 5x + y + 1$ (E) NOTA

21. Suppose A is a $(2x^2 - 10) \times (x^3 + 7x)$ matrix and B is a $(10x^2 - 18) \times (x^3 + 7)$ matrix, both containing only integer entries. Find the sum of all x such that it is possible to compute the product AB.

(A) 11 (B) 10 (C) 9 (D) 8 (E) NOTA

22. Let $\vec{u} = [0, 5, 0]$ and $\vec{v}$ a vector of length 3 that rotates about the xy-plane. Find the maximum length of $\vec{u} \times \vec{v}$.

(A) 15 (B) $\sqrt{34}$ (C) 8 (D) $\frac{25\sqrt{2}}{2}$ (E) NOTA

23. Define $A(p_1, p_2, p_3)$ as the area of the triangle with vertices at p_1, p_2, and p_3. How many of the following statements are true?

I. $A((8, 9), (4, 2), (3, 6)) = A((8/3, 4/3), (3/2, 1/3), (1, 1))$

II. $A((7, 6), (5, 13), (1, 1)) = A((7, 5), (6, 13), (1, 1))$

III. $A((2, 4), (-8, 0), (1, 5)) = A((2, -8), (-4, 32), (1, 1))$

IV. $A((13, 2), (-5, 4), (0, 0)) = A((2, 13), (4, -5), (0, 0))$

(A) I, II, and III only (B) I, II, and IV only

(C) II, III, and IV only (D) I, III, and IV only (E) NOTA

24. Find all values of c such that $[c, 3, 1]$, $[7, 1, -c]$, and $[-6, -2, 2]$ are linearly dependent.

(A) 0, 1 (B) 0, 5 (C) 5 (D) 1, 5 (E) NOTA

25. What is the area of the triangle with side lengths $2\sqrt{10}$, $\sqrt{17}$, and $\sqrt{29}$?

(A) 9 (B) 9.5 (C) 10 (D) 10.5 (E) NOTA

26. Let $\vec{u} = [4e^{4t}\cos 3t - 3e^{4t}\sin 3t, 4e^{4t}\sin 3t + 3e^{4t}\cos 3t]$, and $f(t) = \|\vec{u}\|$. Find the ratio of the maximum value of f to its minimum value, given that $0 \le t \le 5$.

(A) e^{20} (B) e^{10} (C) 10 (D) 5 (E) NOTA

27. Let n be some integer. How many natural numbers d are there such that

$$\frac{\begin{vmatrix} n & -1 \\ 1 & n \end{vmatrix}}{d} \quad \text{and} \quad \frac{\begin{vmatrix} n & -2 \\ 1 & n+2 \end{vmatrix}}{d}$$

are both integers?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

28. Given the vector system

$$\begin{aligned} \vec{x} + \vec{y} + \vec{z} &= [2, 3, 20] \\ 2\vec{x} - \vec{y} + 3\vec{z} &= [7, 27, 37] \\ 3\vec{x} + 2\vec{y} - \vec{z} &= [-6, -10, 20] \end{aligned}$$

Find the value of $\vec{x} \cdot \vec{x} + \vec{y} \cdot \vec{y} + \vec{z} \cdot \vec{z}$.

(A) 118 (B) 211 (C) 304 (D) 397 (E) NOTA

29. Define two sequences a_n and b_n such that $a_1 = 1$ and $a_{n+1} = a_n + 2^{n+1}$ while $b_1 = 1$, $b_2 = 2$, and $b_{n+1} = b_n + 2^n$ for $n \ge 2$. Now form an $n \times n$ matrix X such that $x_{ij} = a_i/b_i$ if $i = j$ and $x_{ij} = 1$ otherwise. As n gets arbitrarily large, $|X|$ approaches what value?

(A) 1 (B) 2 (C) 3 (D) 4 (E) NOTA

30. Which of the following sets constitute a basis for $\mathbb{R}^4$?

(A) {[7,1,2,0], [8,0,1,-1], [1,0,0,-2]}
(B) {[2,17,3,-1], [1,-1,0,3], [5,0,-7,19], [12,0,8,3], [-9,7,2,0]}
(C) {[7,1,2,0], [8,0,1,-1], [1,0,0,-2], [3,0,1,-1]}
(D) {[1,3,2,0], [-2,0,6,7], [0,6,10,7]}
(E) NOTA