2003 Mu Alpha Theta National Convention
Mu Calculus Applications Topic Test – Solutions

1. **(C)**. By the Product Rule, $f'(x) = (x+2)(x+3) + (x+1)(x+3) + (x+1)(x+2)$, hence, $f'(1) = 12 + 8 + 6 = 26$.

2. **(B)**. Using L'Hôpital's Rule, $\lim_{x \to 0} \frac{\sin(5x)}{3x} = \lim_{x \to 0} \frac{5\cos(5x)}{3} = \frac{5}{3}$

3. **(A)**. Average value $= \frac{1}{3-1} \int_1^3 e^x \, dx = \frac{e^3 - e}{2}$

4. **(D)**. The given limit is simply the derivative of f evaluated at $x = 2$. Since $f'(x) = 6x$, the answer is $6(2) = 12$.

5. **(B)**. Choice A and D are false (take $f(x) = |x|$, for example), as well as C because differentiability implies continuity, but that would mean B is true.

6. **(C)**. Note that $2003 \equiv 2004 - 1 \equiv 0 - 1 \equiv 3 \pmod 4$. Since g's derivatives repeat at cycles of 4, the 2003rd derivative is the same as the third derivative, or $-\cos x$.

7. **(A)**. Note that $\sin x + \cos x = \sqrt{2}\sin(x + \pi/4)$. Since $|\sin u| \leq 1$, the maximum is $\sqrt{2}(1) = \sqrt{2}$.

8. **(A)**. By the Product Rule, $y' = 3x^2\cos(4x) - 4x^3\sin(4x)$.

9. **(C)**. Taking the derivative and using a lotta trig identities, the derivative reduces to $3\cos(2x)(\sin(2x) + 1)$ which, when evaluated at $x = \pi/4$, gives 0.

10. **(B)**. Differentiating implicity, we get $3x^2 + 3y^2y' = ay + axy'$, and after some rearrangement, $y' = -(3x^2 - ay)/(3y^2 - ax)$. Since $x = y = 2003$, y' is just -1.

11. **(D)**. Simplify first to get $x(t) = t^6 + 3t^5 + t^4$, differentiate twice to get $x''(t) = a(t) = 12t^2 + 60t^3 + 30t^4$, and $a(1) = 102$.

12. **(A)**. First, we find the value of R, which is $((100)(200))/(100 + 200) = 200/3$. Then, differentiating the formula with respect to t, we get $R'/R^2 = R_1'/R_1^2 + R_2'/R_2^2$. Thus, $R' = (200/3)^2(.4/(100)^2 + .5/(200)^2) = 7/30$.

13. **(A)**. If θ is the angle between the top of the ladder and wall and x is the distance from the wall to the base of the ladder, then the two variables can be related by $\sin\theta = x/15$. Differentiating, we get $(15\cos\theta)\theta' = dx$. Plug in values to get $\theta' = \sqrt{2}/5$.

14. **(C)**. Note that f is nonnegative on the given interval. Thus, $A(c) = \int_0^c x^4 + 2x^2 + 12$, or $A'(c) = c^4 + 2c^2 + 12$, making $A'(5) = 687$.

15. **(E)**. Writing out the first few terms to try to find a pattern, we find that the kth partial sum is $k/(k+1)$. The k in this problem is 2003, so the answer is 2003/2004.

16. **(E)**. Working out the first few terms reveals that $a_n = 1/(10^{2003} + n - 1)$. Since the sum of a harmonic series diverges, the given sum has no upper bound.

17. **(B)**. By L'Hôpital's Rule, $\displaystyle\lim_{x\to\pi/2} \frac{e^{\cos x} - 1}{x - \pi/2} = \lim_{x\to\pi/2} \frac{-(\sin x)e^{\cos x}}{1} = -1$

18. **(A)**. Since $f(0) = 0$ and $f(5) = -4$, f has a root in-between that interval by the Intermediate Value Theorem. It's easy to check that the other intervals don't contain a root in them.

19. **(D)**. The area is given by $A(x) = 2xy = 2x(64-4x^2) = 128x - 8x^3$. Setting $A'(x) = 0$ yields a critical value of $x = 4/\sqrt{3}$, which is a maximum by the First Derivative Test. The answer is $A(4/\sqrt{3}) = 1024\sqrt{3}/9$.

20. **(C)**. By the matrix formula for the area of a triangle given three vertices, we get that the area expression in terms of x (in fact, it's actually independent of $f(x)$!) is given by $5|x|$, making the maximum $5(20) = 100$.

21. **(B)**. The problem stated in the language of differential equations becomes $f'(x) = f(x)$, or after separation of variables, $f(x) = Ce^x$ for some constant C. Plug in the initial condition and get that $C = 3/e^5$, making $f(x) = 3e^{x-5}$, so $f(6) = 3e \approx 8.15$.

22. **(D)**. Set the y-values equal to each other to obtain intersection points of $(0,0)$, $(-1,-1)$, and $(1,1)$. Using symmetry and the Shell Method, the volume is $2\left(2\pi\int_0^1 x(x-x^3)\,dx\right) = 8\pi/15$.

23. **(A)**. In order for the two curves to be tangent to each other, they need to intersect and have equal derivatives at that intersection point. Solving $x^3 - 3x + 4 = 3(x^2 - x)$ and $3x^2 - 3 = 3(2x - 1)$ yields a common value of $x = 2$, so the point is $(2, 6)$.

24. **(B)**. The distance from the axis of revolution to the center of the circle is 6 while its area is π. By the Theorem of Pappus, the volume is $2\pi(6)(\pi) = 12\pi^2$.

25. **(A)**. The ellipse has a larger radius of 5 and smaller diameter of 3. By the standard formula the area is $\pi(5)(3) = 15\pi$.

26. **(B)**. The graphs intersect at $(0,0)$ and $(10,100)$, as easily shown by setting the equations equal to each other. The area is then $\int_0^{10} 10x - x^2\,dx = 500/3$.

27. **(D)**. Integration is the operation needed to sum up the cross sections. The volume is $\int_0^{10} 3x^2 + 3\,dx = 1030$.

28. **(B)**. Let the limit equal L. Take natural logs of both sides and write the limit as $\lim_{x\to\infty}(\ln(2^x - 1)/x - (\ln x)/x) = \ln L$. Using L'Hôpital's Rule separately, the left-hand side is equal to $\ln 2 - 0 = \ln 2$. Thus, $L = e^{\ln 2} = 2$.

29. **(A)**. Let $x = 0$ in the inequality to get $(P(0))^2 + 4 \leq 4P(0)$, or $(P(0) - 2)^2 \leq 0$. The square of any quantity can't be negative so we can conclude that $P(0) = 2$. Similarly, letting $x = 1$ yields $P(1) = 2$. Since $P(0) = P(1)$, there is a number c in between 0 and 1 such that $P'(c) = 0$ (Rolle's Theorem), contradicting the fact that $P'(x) > 0$ for all x. No such polynomials P exist.

30. **(A)**. Let $A = (0, 0)$, $B = (2, 0)$, and $C = (0, \sqrt{3})$. The point P that minimizes the total distance is called the *Fermat Point* and is obtained by erecting equilateral triangles on each side of ABC and finding the intersection point of each of the lines passing through a vertex of ABC and the farthest equilateral triangle vertex opposite to it. After a lengthy calculation, we find that $P = (5/13, 3\sqrt{3}/13)$, making the answer A.