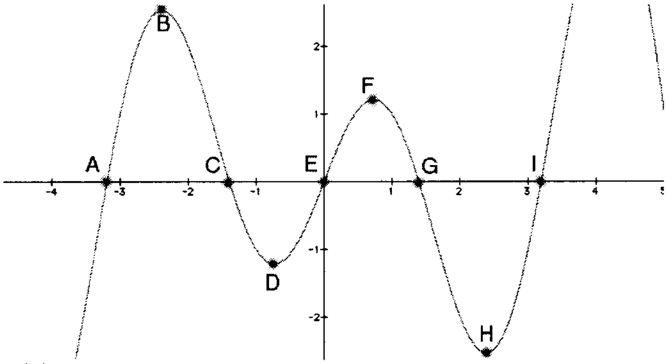
Question #1
Calculus Bowl
Mu Alpha Theta National Convention 2003

The graph below is a plot of $y = f'(x)$. How many of the statements below the graph are always true?



I. A relative maximum of $f'(x)$ exists at point A.

II. $f(x)$ is concave down at point B.

III. $f(x)$ is concave up at point C.

IV. $f''(x)$ at the point D is zero.

V. $f(x)$ has an inflection point at point E.

VI. A relative maximum of $f'(x)$ exists at point F.

VII. A relative minimum of $f(x)$ exists at point G.

VIII. A relative minimum of $f(x)$ exists at point I.

Question #2
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If $f(x) = (4x^2+1)^2(2x-7)^3\cos^3\left(x - \frac{\pi}{4}\right)$, then what is the value of $f'(0)$?

Question #3
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Let *A*, *B*, *C*, and *D* (the constants in the following problems) all be natural numbers.

$$\int_1^A \sqrt{w-1}\,dw = 144 \qquad \int_{B/2}^{B} x^2 + 3x + 4\,dx = 690$$

$$\int_1^C \frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2} = \frac{2}{5} \qquad \int_1^D \frac{(\log_5 z)^2}{z}\,dz = 9$$

What is the sum of *A*, *B*, *C*, and *D*?

Question #4
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At noon, the sailboat *Mu* was 12 nautical miles due north of the sailboat *Alpha*, whereas the sailboat *Theta* was 8 miles due east of the *Alpha*. The *Mu* was sailing south at 12 knots (nautical miles per hour) and continued to do so all day. The *Alpha* was sailing northeast at $8\sqrt{2}$ knots and continued to do so all day. The *Theta* was sailing west at 10 knots and continued to do so all day. Let *A* be the length of time (in minutes, to the nearest minute) when the *Mu* and the *Alpha* are closest together; let *B* be the length of time (in minutes, to the nearest minute) when the *Mu* and the *Theta* are closest together; and let *C* be the length of time (in minutes, to the nearest minute) when the *Alpha* and the *Theta* are closest together. What is the sum of *A*, *B*, and *C*?

Question #5
Calculus Bowl
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Evaluate the following definite integral: $\int_{-4}^{4}\left|x^3-7x-6\right|dx$

Question #6
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This question is a relay type question. The answer to part (I), A, will be used in part (II)...and so on through part (III). **On your answer sheet, put down the exact value of C.**

(I). $\int_0^{2\pi} x\cos\left(x/2\right)dx = A$

(II). $\lim_{y\to 3}\dfrac{y^2+2y-15}{y^2-8y+A+23} = B$

(III). Let $f(z) = z^2\cos\left(z/B\right)$. $f'(\pi) = C$

Question #7
Calculus Bowl
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What is $A+16B+\dfrac{C\sqrt{2}}{2}+3D$ if...

$A = f(4)$ for $\int_0^w f(t)dt = w\cos(\pi w)$, $B = g(4)$ for $\int_0^{x^2} g(t)dt = x\cos(\pi x)$,

$C = h(2)$ for $h(y) = \dfrac{d}{dy}\left(\int_{-2y}^{y^2} t^3\sqrt{4+t}\,dt\right)$, and $D = \lim_{z\to 0}\dfrac{\int_0^{z^2}\frac{t^2}{t^4+1}dt}{z^6}$?

Question #8
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A wire of length L is cut into two pieces, one being bent to form a square and the other to form an equilateral triangle. Let A be the length of the wire piece used to form the square if the sum of the two areas formed is a minimum. Let B be the length of the wire piece used to form the square if the sum of the two areas formed is a maximum. What is the sum of A and B? (Note: Allow the possibility that the wire may not be cut at all, and only one shape is formed from the entire wire.)

Question #9
Calculus Bowl
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What is $AB+CD$, if:

$A = \lim_{w\to 0}\tan 2w\csc 4w$

$B = \lim_{x\to 6}\dfrac{x^4-2x^3-23x^2-12x+36}{x^4-6x^3-5x^2-106x+120}$

$C = \lim_{y\to\infty}\left(3y^3+5\right)^{1/\sqrt[3]{y}}$

$D = \lim_{z\to -\infty}\sqrt{z^2-5z}-\sqrt{z^2+7z}$

Question #10
Calculus Bowl
Mu Alpha Theta National Convention 2003

Of the four statements or expressions below, which one(s) is/are true? Mark the letters for the true responses on your answer sheet.

A. Calculating $\int_1^3 x^2dx$ using the trapezoid rule with 6 subintervals on [1,3] gives a value of $235/27$.

B. $\int_0^{\pi/4}\tan^4 x\sec^4 x\,dx = \dfrac{11}{35}$

C. The volume generated when the region bounded by the circle $y^2+z^2-10y+21=0$ in the yz-plane is revolved about the z-axis (to make a torus) is 40π.

D. The length of the curve $5y^3 = x^2$ that lies inside the circle $x^2+y^2=6$ is $134/27$.

Question #11
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For each section below, let g be the inverse function of f:

(I.) $f(x)=1+1/x$. $g'(3)=A$.

(II.) $f(x)=x^5+x^3$. $g'(2)=B$.

(III.) $f(x)=x^2-4x-3$, $x>2$. $g'(2)=C$.

(IV.) $f(x)=\ln x$. $g'(1)=D$.

What is the value of $(1/A)+(1/B)+(1/C)+(1/D)$?

Question #12
Calculus Bowl
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The curve whose derivative is $dy/dx=\sqrt{3y-7x+xy-21}$ passes through the points (-2, 8), (1, A), (6, B), (13, C), and (22, D). What is the sum of A, B, C, and D ?

Question #13
Calculus Bowl
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If...

$$A=\int_0^{\sqrt{3}} x^3\sqrt{1+x^2}\,dx,\ B=\int_0^{\sqrt{\pi/2}} x\tan^2\left(x^2\right)dx,$$

$$C=\int_3^5 \frac{3x^2-6x-28}{x^3-3x^2-28x+60}\,dx, \text{ and } D=\int_1^e x(\ln x)^2\,dx,$$

then what is the value of $A-2B+e^C+D-\dfrac{58}{15}$?

Question #14
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The legs of an isosceles triangle have length 12 each. The vertex angle (the angle opposite the base) is decreasing at a rate of 0.5 radians per second. The parameter A is equal to the rate of change of the area of the triangle when the vertex angle is $\pi/3$ radians. The parameter B is equal to the rate of change of the perimeter of the triangle when the vertex angle is $\pi/3$ radians. What is $|A|+|B|$ (ignore the difference between length and area units)?

Question #15
Calculus Bowl
Mu Alpha Theta National Convention 2003

If $f(x)=\dfrac{4x}{\sqrt{8x^2+1}}$, then what is the value of the expression below?

$$\left[f\left(\sqrt{3}\right)\right]^2+25f'\left(-\sqrt{3}\right)+\lim_{x\to-\infty}f(x)+\int_0^{\sqrt{3}} f(x)\sqrt{8x^2+1}\,dx$$

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The correct answer to each question is given immediately after the question number in parentheses.

1. (3) I is false because $f'(x)$ is rising at A. II is false because $f''(x)$ is zero at B. III is false because the slope of $f'(x)$ is negat at C, so $f(x)$ is concave down. IV is true. V is false because $f''(x)$ is positive at D, not zero. VI is true. VII is false because the slope of $f'(x)$ is negative at G, implying that $f(x)$ has a relative maximum there. VIII is true. So there are three true statements.

2. $\left(-735\sqrt{2}/4\right)$ $f'(x)=2(8x)(4x^2+1)(2x-7)^3\cos^3(x-\pi/4)+3(2)(4x^2+1)^2(2x-7)^2\cos^3(x-\pi/4)-3\sin(x-\pi/4)(4x^2+1)^2(2x-7)^3\cos^2(x-\pi/4$

$f'(0)=0+6(1)(49)\left(\sqrt{2}/2\right)^3-3\left(-\sqrt{2}/2\right)(1)(-343)\left(\sqrt{2}/2\right)^2=-735\sqrt{2}/4$

3. (255) $\int_1^A\sqrt{w-1}\,dw=144=\left(\frac{2}{3}(w-1)^{3/2}\right]_1^A\rightarrow A=37$; $\int_{B/2}^{B}x^2+3x+4\,dx=690=\left(\frac{x^3}{3}+\frac{3x^2}{2}+4x\right]_{B/2}^{B}=\frac{7B^3}{24}+\frac{9B^2}{8}+2B\rightarrow B=12$;

$\int_1^C\frac{dy}{2\sqrt{y}\left(1+\sqrt{y}\right)^2}=\frac{2}{5}=\left(\frac{-1}{1+\sqrt{y}}\right]_1^C\rightarrow C=81$; and $\int_1^D(\log_5 z)^2\,dz/z=9\ln 5=\left[1/(\ln 5)^2\right]\int_1^D(\ln z)^2\,dz/z=\left[1/(\ln 5)^2\right]\left[(\ln z)^3/3\right]_1^D\rightarrow D=125.$

So the sum of the four numbers is 255.

4. (108) Positions: M, $x=0$ and $y=12-12t$; A, $x=8t$ and $y=8t$; Th, $x=8-10t$ and $y=0$

Distance: MA, $s^2=(8t)^2+(20t-12)^2\rightarrow 2s\,ds/dt=128t+800t-480=0\rightarrow t=15/29$ (hrs) ≈ 31 min $=A$,

Mθ, $s^2=(8-10t)^2+(12-12t)^2\rightarrow 2s\,ds/dt=160-200t+288-288t=0\rightarrow t=56/61$ (hrs) ≈ 55 min $=B$, and

Aθ, $s^2=(18t-8)^2+(8t)^2\rightarrow 2s\,ds/dt=648t-288+128t=0\rightarrow t=36/97$ (hrs) ≈ 22 min $=C$. So the sum of A, B, and C is 108.

5. (76) $\int_{-4}^{4}\left|x^3-7x-6\right|dx=\int_{-4}^{-2}-(x^3-7x-6)dx+\int_{-2}^{-1}(x^3-7x-6)dx+\int_{-1}^{3}-(x^3-7x-6)dx+\int_{3}^{4}(x^3-7x-6)dx$

$=\left(-x^4/4+7x^2/2+6x\right]_{-4}^{-2}{}_{-1}^{3}+\left(x^4/4-7x^2/2-6x\right]_{-2}^{-1}{}_{3}^{4}=76$

6. $\left(-\pi^2\sqrt{2}/8+\pi\sqrt{2}\right)$ $\int_0^{2\pi}x\cos(x/2)dx=\left(2x\sin(x/2)+4\cos(x/2)\right]_0^{2\pi}=-8=A$, $\lim_{y\to 3}\frac{y^2+2y-15}{y^2-8y+15}=\lim_{y\to 3}\frac{y+5}{y-5}=-4=B$, and

$f(z)=z^2\cos(-z/4)\rightarrow f'(z)=z^2\left[-\sin(-z/4)\right](-1/4)+2z\cos(-z/4)\rightarrow f'(\pi)=-\pi^2\sqrt{2}/8+\pi\sqrt{2}=C$

7. (518) $(d/dw)\left[\int_0^w f(t)dt=w\cos(\pi w)\right]\rightarrow f(w)=-\pi w\sin(\pi w)+\cos(\pi w)\rightarrow f(4)=1=A.$ $(d/dx)\left[\int_0^{x^2}g(t)dt=x\cos(\pi x)\right]=2xf(x^2)=$

$-\pi x\sin(\pi x)+\cos(\pi x)\rightarrow f(2^2)=1/4=B.$ $h(y)=(d/dy)\left(\int_{-2y}^{y^2}t^3\sqrt{4+t}\,dt\right)=(2y)y^6\sqrt{4+y^2}-(-2)(-8y^3)\sqrt{4-2y}.$ $h(2)=512\sqrt{2}=C.$

$\lim_{z\to 0}\frac{\int_0^{z^2}t^2\,dt/(t^4+1)}{z^6}=\frac{0}{0}$. Use L'Hopital's rule, $\lim_{z\to 0}\frac{2z^5/(z^8+1)}{6z^5}=\lim_{z\to 0}\frac{1}{3(z^8+1)}=\frac{1}{3}=D$ So $A+16B+C\sqrt{2}/2+3D=1+4+512+1=518.$

8. $\left[\left(12\sqrt{3}-5\right)L/11\right]$ Let x be a side of the square and y be a side of the triangle. Then $4x+3y=L\rightarrow y=(L-4x)/3$.

$Area=x^2+y^2\sqrt{3}/4=x^2+\left(\sqrt{3}/4\right)\left[(L-4x)/3\right]^2\rightarrow Ared'=2x+2\left[(L-4x)/3\right](-4/3)\left(\sqrt{3}/4\right)=0\rightarrow x=\left(3\sqrt{3}-4\right)L/11.$

$Ared''(x)=2+8\sqrt{3}/9>0\rightarrow$ minimum area $\rightarrow 4x=\left(12\sqrt{3}-16\right)L/11=A$. The maximum area forms when all of the area is in the square

Then the area is $(L/4)^2=L^2/16$ (as compared to $L^2\sqrt{3}/36$ for the triangle). $\therefore B=L,\ A+B=\left(12\sqrt{3}-5\right)L/11.$

9. (6) $\lim_{w\to 0}\tan 2w\csc 4w=\lim_{w\to 0}\frac{\sin 2w}{\cos 2w\sin 4w}=\lim_{w\to 0}\frac{2\cos 2w}{4\cos 2w\cos 4w-2\sin 2w\sin 4w}=\frac{1}{2}=A.$

$\lim_{x\to 6}\frac{x^4-2x^3-23x^2-12x+36}{x^4-6x^3-5x^2-106x+120}=\frac{0}{-696}=0=B.$ $C=\lim_{y\to\infty}(3y^3+5)^{1/\sqrt[3]{y}}\rightarrow\ln C=\lim_{y\to\infty}\frac{\ln(3y^3+5)}{y^{1/3}}=\lim_{y\to\infty}\frac{9y^2/(3y^3+5)}{y^{-2/3}/3}=\lim_{y\to\infty}\frac{27y^{8/3}}{3y^3+5}=0\rightarrow C=1.$

$\lim_{z\to-\infty}\sqrt{z^2-5z}-\sqrt{z^2+7z}=\lim_{z\to-\infty}\left(\sqrt{z^2-5z}-\sqrt{z^2+7z}\right)\left(\frac{\sqrt{z^2-5z}+\sqrt{z^2+7z}}{\sqrt{z^2-5z}+\sqrt{z^2+7z}}\right)=\lim_{z\to-\infty}\frac{-12z}{\sqrt{z^2-5z}+\sqrt{z^2+7z}}=6=D.$ So, $AB+CD=6.$

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10. (A,D) $T=(h/2)(y_0+2y_1+2y_2+2y_3+2y_4+2y_5+y_6)=(1/6)(1+32/9+50/9+8+49/9+64/9+9)=235/27$., TRUE. $\int_0^{\pi/4}\tan^4 x\sec^4 x\,dx=\int_0^{\pi/4}\tan^4 x(\tan^2 x+1)\sec^2 x\,dx=\left(\tan^7 x/7+\tan^5 x/5\right]_0^{\pi/4}=12/35$, which means B is FALSE. The volume of a torus is the cross-sectional area times the length. In this case, the circle's area is 4π. The length is 10π. So the volume is $40\pi^2$, which means the C statement is FALSE. $5y^3=x^2$ intersects the circle $x^2+y^2=6$ where $5y^3=6-y^2\rightarrow 5y^3+y^2-6=0$, which has a root at $y=1$. The variable x has two roots at $\pm\sqrt{5}$. From our original formula, we know that $dx/dy=3\sqrt{5y}/2$, so the arc length is $s=2\int_0^1\sqrt{1+(45y/4)}\,dy=\left((16/135)[1+(45y/4)]^{3/2}\right]_0^1=134/27$, which is TRUE.

11. $(10+1/e)$ $g(f(x))=x\rightarrow g'(f(x))f'(x)=1\rightarrow g'(f(x))=1/f'(x)$. For (I.), $f(x)=3$ when $x=1/2$. $f'(x)=-x^{-2}$ $\rightarrow g'(3)=1/f'(1/2)=-1/4=A$. For (II.), $f(x)=2$ when $x=1$. $f'(x)=5x^4+3x^2\rightarrow g'(2)=1/f'(1)=1/8=B$. For (III.), $f(x)=2$. when $x=5$ (since $x>2$). $f'(x)=2x-4\rightarrow g'(2)=1/f'(5)=1/6=C$. For (IV.), $f(x)=1$ when $x=e$. $f'(x)=x^{-1}\rightarrow g'(1)=1/f'(e)=e=D$. So $1/A+1/B+1/C+1/D=-4+8+6+1/e=10+1/e$.

12. (7226/3) $dy/dx=\sqrt{(x+3)(y-7)}\rightarrow dy/\sqrt{y-7}=\sqrt{x+3}\,dx\rightarrow 2\sqrt{y-7}=(2/3)(x+3)^{3/2}+C$. Plugging in (-2,8) gets a C=4/3. Plugging in 1, 2, 3, and 4 for x gets 163/9, 904/9, 4419/9, and 16192/9, which has a sum of 7226/3.

$\int_1^4(u-1)\sqrt{u}/2\,du=\left(u^{5/2}/5-2u^{3/2}/3\right]_1^4=58/15$

13. $\left[(e^2+\pi)/4\right]$ Let $u=1+x^2$, then $du=2x\,dx$, and we have $\int_1^4(u-1)\sqrt{u}/2\,du=\left(u^{5/2}/5-2u^{3/2}/3\right]_1^4=58/15=A$. Let $u=x^2$, then $du=2x\,dx$ and $\int_0^{\pi/4}\frac{1}{2}\tan^2 u\,du=\int_0^{\pi/4}\frac{1}{2}(\sec^2 u-1)\,du=\left((\tan u)/2-u/2\right]_0^{\pi/4}=(1/2)-(\pi/8)=B$. $\int_3^5\frac{3x^2-6x-28}{x^3-3x^2-28x+60}dx=\int_3^5\frac{1}{x+5}+\frac{1}{x-2}+\frac{1}{x-6}dx$ $=\left(\ln|x+5|+\ln|x-2|+\ln|x-6|\right]_3^5=\ln\frac{5}{4}=C$. Let $u=(\ln x)^2$ and $dv=x\,dx$. Then $du=2\ln x\,dx/x$ and $v=x^2/2$. This makes the integral equal to $x^2(\ln x)^2/2-\int x\ln x\,dx$. Another int. by parts with $u=\ln x$ and $dv=x\,dx$ gives $du=dx/x$ and $v=x^2/2$. This makes the integral $x^2(\ln x)^2/2-(x^2\ln x)/2+\int x\,dx/2$. Solving for the definite integral gives $\left(x^2(\ln x)^2/2-(x^2\ln x)/2+x^2/4\right]_1^e=(e^2-1)/4$. Solving for what the question asks for: $58/15-1+\pi/4+e^{\ln(5/4)}+(e^2-1)/4-58/15=(e^2+\pi)/4$.

14. $(18+3\sqrt{3})$ $Area=(1/2)\cdot 12\cdot 12\cdot\sin\theta=72\sin\theta\rightarrow dArea=72\cos\theta\,d\theta=72(1/2)(-1/2)=-18=A$. $Per.=24+12\sqrt{2(1-\cos\theta)}\rightarrow dPer.=12\sqrt{2}\sin\theta\,d\theta/\left(2\sqrt{1-\cos\theta}\right)=6\sqrt{2}\left(\sqrt{3}/2\right)(-1/2)/\sqrt{1/2}=-3\sqrt{3}=B$. So $|A|+|B|=18+3\sqrt{3}$.

15. $\left[(218/25)-\sqrt{2}\right]$ $f(\sqrt{3})=4\sqrt{3}/\sqrt{8(\sqrt{3})^2+1}=4\sqrt{3}/5\rightarrow\left[f(\sqrt{3})\right]^2=48/25$.
$f(x)=4x(8x^2+1)^{-1/2}$, so $f'(x)=4x(-1/2)(8x^2+1)^{-3/2}(16x)+4(8x^2+1)^{-1/2}\rightarrow f'(-\sqrt{3})=(-96/125)+(4/5)\rightarrow 25f'(-\sqrt{3})=4/5$
$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}1/\sqrt{(8x^2+1)/16x^2}=\lim_{x\to-\infty}1/\sqrt{(1/2)+(1/16x^2)}=\sqrt{2}$, but remember we're going to -∞, so the answer is $-\sqrt{2}$.
$\int_0^{\sqrt{3}}4x\,dx=\left(2x^2\right]_0^{\sqrt{3}}=6$. Putting it all together gets $(218/25)-\sqrt{2}$.