Mu Alpha Theta National Convention 2003
Calculus Individual Test

For all questions, answer E. "NOTA" means none of the above answers is correct.

1. The total cost, c, to a company for selling x "Barker's Backyard Barbeque Grills" is $c(x) = x^2 + 4x + 100$. The selling price per grill is $p(x) = 100 - x$. What price per grill will yield the maximum profit for the company?

A. 98 B. 76 C. 50 D. 24 E. NOTA

2. For $x > 0$, $\int \frac{2x + x^2 - \sqrt{x}}{x^2}\,dx =$

A. $\frac{2 + 2\sqrt{x}\,\ln x + x\sqrt{x}}{\sqrt{x}} + C$ B. $\frac{1 + 4\sqrt{x}\,\ln x + 2x\sqrt{x}}{2\sqrt{x}} + C$

C. $\frac{2x^2 + 4\sqrt{x} + x^3\sqrt{x}}{x^2\sqrt{x}} + C$ D. $\frac{2 + 2\sqrt{x}\,\ln x}{\sqrt{x}} + C$ E. NOTA

3. Find the slope of the curve $y = 8 - x - x^2$ at its second quadrant point of intersection with the line $x - 2y + 7 = 0$.

A. −4 B. −2 C. 2 D. 5 E. NOTA

4. The radius of a circle increases from 2.00 cm to 2.02 cm. Use differentials to estimate the resulting change in the circle's area.

A. $\frac{2\pi}{25}$ B. $\frac{4\pi}{25}$ C. 4π D. $\frac{\pi}{25}$ E. NOTA

5. Use the Trapezoidal Rule with $n = 3$ to approximate the area under the curve $y = \cos x$ and above the x-axis from $a = 0$ to $b = \frac{\pi}{2}$.

A. $\frac{\pi(2 + \sqrt{3})}{12}$ B. $\frac{\pi(1 + \sqrt{3})}{12}$ C. $\frac{\pi(1 + \sqrt{3})}{6}$ D. $\frac{\pi(2 + \sqrt{3})}{6}$ E. NOTA

6. Consider the function $g(x) = 3x^2 e^x$. How many of the following statements regarding g(x) is/are true?

I. g(x) is positive for all real values of x.
II. g(x) is decreasing on the interval (0,2).
III. g(x) has exactly one point of inflection.
IV. g(x) has a relative minimum at $x = -2$.

A. 1 B. 2 C. 3 D. 4 E. NOTA

7. Let $u(x) = \sqrt{x^2 + 9}$ and $v(x) = 3x^3 - 2x$. Find $\frac{du}{dv}$ evaluated at $x = 4$.

A. $\frac{1}{1420}$ B. $\frac{3}{755}$ C. $\frac{4}{855}$ D. $\frac{2}{355}$ E. NOTA

8. Let f be a function such that $f(x + h) - f(x) = 6xh + 3h^2$ and $f(1) = 5$. Determine $f(2) + f'(2)$.

A. 8 B. 13 C. 16 D. 26 E. NOTA

9. If $f(x) = \ln(2x + 3)$ for $x > -\frac{3}{2}$, then the nth derivative of f at x equals

A. $\dfrac{-(2)^n (n-1)!}{(2x+3)^n}$ B. $\dfrac{(-1)^{n+1}(n-1)!}{(2x+3)^n}$ C. $\dfrac{-(-2)^n (n-1)!}{(2x+3)^n}$ D. $\dfrac{2^n}{(2x+3)^n}$ E. NOTA

10. An equation of the tangent line to the graph of a differentiable function g at x = 0 is $y + 1 = 4(x - 0)$. Evaluate $\lim\limits_{x \to 0} \dfrac{\sin(3x)}{2x\, g(x)}$.

A. $-\frac{3}{2}$ B. $\frac{1}{8}$ C. $\frac{3}{8}$ D. $\frac{1}{2}$ E. NOTA

11. The value of a particular investment at time t is increasing at a rate proportional to the difference between \$20,000 and its value at t. At t = 0, the value is \$2000 while at t = 1, the value is \$3000. Find the value (in dollars) at t = 3.

A. $20{,}000 - 17{,}000(\frac{17}{18})^2$ B. $20{,}000 - 18{,}000(\frac{17}{18})^2$
C. $20{,}000 - 17{,}000(\frac{17}{18})^3$ D. $20{,}000 - 18{,}000(\frac{17}{18})^3$ E. NOTA

12. Let $g(x) = \int_2^{3x} \sin^2 t\, dt$. Find $g'''(x)$.

A. $54\cos(6x)$ B. $-162\cos(3x)\sin(3x)$
C. $-18\sin^2(3x) + 18\cos^2(3x)$ D. $-216\sin(3x)\cos(3x)$ E. NOTA

13. Let s be the length of each one of the congruent sides of an isosceles triangle and let θ be the angle between them. If s is decreasing at the rate of $\frac{1}{10}$ ft/min and θ is increasing at the rate of $\frac{\pi}{90}$ radians/min, then at what rate, in ft²/min, is the area of the triangle changing when $s = 2\sqrt{3}$ ft and $\theta = \frac{\pi}{6}$?

A. $\dfrac{\sqrt{3}\,\pi}{30} + \dfrac{\sqrt{3}}{10}$ B. $\dfrac{1}{60}(\pi + 6\sqrt{3})$ C. $\dfrac{1}{30}(\pi - 9)$ D. $\dfrac{\sqrt{3}}{30}(\pi - 3)$ E. NOTA

14. Find $\dfrac{dy}{dx}$ for the relation $\sin x = x + x\tan y$

A. $\dfrac{1 + \tan y - \cos x}{x\sec^2 y}$ B. $\dfrac{\cos x - 1 - \tan y}{x + x\sec^2 y}$ C. $\dfrac{x\cos x - \sin x + x}{x^2\sec^2 y}$ D. $\dfrac{x\sin x - \cos x}{x^2 + x^2\sec^2 y}$ E. NOTA

15. The function $a(t) = (t-8)^{-2/3}$ for $t \geq 0$ represents the acceleration (in ft/sec^2) of a moving body whose velocity after 9 seconds is 6 ft/sec. Find the total distance traveled (in feet) by the body during the first 8 seconds.

A. 12 B. $\frac{27}{2}$ C. 36 D. 288 E. NOTA

16. Given that $f''(x) = 6|x|$, which of the following functions could be f(x)?

I. $|x^3|$ II. $x \cdot |x^2|$ III. $\frac{|x^4|}{x}$

IV. $|x| \cdot x^2$ V. $\frac{x^4}{|x|}$

A. I, II & III only B. I & IV only C. I, IV & V only D. II, III & V only E. NOTA

17. The "Mean Value Theorem for Integrals" states that for a continuous function f(x) on the interval [a,b], there is at least one value c in the interval for which f(c) is equal to the average value of the function on the interval. Given the function $f(x) = \begin{cases} x^3 \text{ if } 0 \leq x \leq 1 \\ x^2 \text{ if } 1 < x \leq 3 \end{cases}$ on the interval [0,3], find all values of c guaranteed by the theorem.

A. $\sqrt[3]{\frac{107}{36}}$ only B. $\sqrt[3]{\frac{107}{12}}$ only C. $\frac{\sqrt{107}}{6}$ only D. $\frac{\sqrt{107}}{2\sqrt{3}}$ only E. NOTA

18. Given $h(x) = x^2$, $g(x) = h(1 + f(x))$ and $f'(1) = g'(1) = 1$. Find f(1).

A. −1 B. $-\frac{1}{2}$ C. 0 D. 1 E. NOTA

19. Let R_1 be the plane region bounded by the curves $y = \sqrt{x^3}$, $y = 0$ and $x = 4$. Let R_2 be the plane region bounded by the curves $y = \sqrt{x^3}$, $x = 0$ and $y = 8$. Let V_1 be the volume of the solid formed when R_1 is revolved about the x-axis. Let V_2 be the volume of the solid formed when R_2 is revolved about the x-axis. Find the ratio of V_1 to V_2.

A. 1 : 2 B. 1 : 3 C. 1 : 4 D. 1 : 6 E. NOTA

20. To which of the following functions does Rolle's Theorem apply?

A. $f(x) = \frac{x^2 - 1}{x}$ on [−1,1] B. $g(x) = 4 + |x-2|$ on [0,4]

C. $h(x) = x^2 - 2x$ on [−2,0] D. $k(x) = 4 - \frac{3}{x}$ on [1,3] E. NOTA

21. Evaluate $\int \frac{5x\,dx}{\sqrt[3]{(x^2+7)^2}}$

A. $\frac{5}{6}\sqrt[3]{x^2+7}+C$
B. $\frac{3}{10}\sqrt[3]{x^2+7}+C$
C. $\frac{15}{2}\sqrt[3]{x^2+7}+C$
D. $\frac{1}{30}\sqrt[3]{x^2+7}+C$
E. NOTA

22. An index of consumer confidence fluctuates between −1 and 1. Over a two-year period, beginning at time $t = 0$, the level of this index, c, is $c(t) = \frac{t\cos(t^2)}{2}$, where t is measured in years. Find the average value of this index over the two-year period.

A. $-\frac{1}{4}\sin(4)$
B. 0
C. $\frac{1}{4}\sin(4)$
D. $\frac{1}{2}\sin(4)$
E. NOTA

23. Suppose f is continuous and $f(x) = 1 + \int_0^x [(f(t))^2 + 1]\,dt$ for $0 \le x \le \frac{1}{2}$. Which of the following functions could be f(x) on $[0, \frac{1}{2}]$?

I. $\frac{1}{x^2+1}$
II. $\arctan(x) + 1$
III. $\tan(x) + 1$
IV. $\tan(x + \frac{\pi}{4})$

A. I & IV only
B. II & III only
C. III only
D. IV only
E. NOTA

24. Find the equation of the line joining the inflection points of $y = \frac{2-x}{x^2+4}$.

A. $3x + 16y = 2$
B. $x + 16y = 6$
C. $3x + 8y = 4$
D. $x + 8y = 12$
E. NOTA

25. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2}$

A. ln 2
B. $\frac{\sqrt{3}}{2}$
C. $\frac{\pi}{4}$
D. 1
E. NOTA

26. Let f(x) equal the function whose derivative at any value of x is $f'(x) = -2x\,e^{-x^2}$ and satisfying the condition $f(0) = 1$.

Let g(x) equal $\lim_{h \to 0} \dfrac{\int_1^{x+h} e^{-t^2}\,dt - \int_1^{x} e^{-t^2}\,dt}{h}$.

Let h(x) equal $\dfrac{d}{dx}\left(\dfrac{-e^{-x^2}}{2x}\right)$.

Which of these functions are equal?

A. f and g only B. g and h only C. f and h only D. f, g and h E. NOTA

27. Let $f(x) = \begin{cases} 3x^2 & \text{for } 0 \le x \le 1 \\ 4 - x & \text{for } 1 < x \le 4 \end{cases}$ and let R be the region bounded by the graph of f, the x-axis, and the lines $x = b$ and $x = b + 2$, where $0 \le b \le 1$. Determine the value of b that maximizes the area of R.

A. $\frac{1}{2}$ B. $\frac{2}{3}$ C. $\frac{3}{4}$ D. 1 E. NOTA

28. A disease is spreading through a population in a manner that can be modeled by the function $f(t) = \dfrac{P}{1 + 3e^{-t}}$ where P is the total population and f(t) is the number infected at time t. What proportion of the population is infected when the disease is spreading the fastest?

A. $\frac{1}{3}$ B. $\frac{1}{e}$ C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. NOTA

29. Which one of the following is an equation of the line with slope $\frac{25}{2}$ that is normal to the graph of $y = \dfrac{1}{(5x + 2)^2}$?

A. $625x - 50y = 373$ B. $625x - 50y = 371$

C. $625x - 50y = 367$ D. $625x - 50y = 361$ E. NOTA

30. Evaluate $\int_{1/2}^{1} 3x\sqrt{2x - 1}\,dx$

A. $\frac{233}{960}$ B. $\frac{4}{15}$ C. $\frac{233}{320}$ D. $\frac{4}{5}$ E. NOTA

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① Profit = revenue − cost
$= x(100-x) - (x^2+4x+100)$
$= -2x^2 + 96x - 100$ which is maximized when $-4x+96=0$ or $x = 24$; price is $100-24 = 76 \to$ B/

② $\int \left(\frac{2}{x} + 1 - x^{-3/2}\right) dx$
$= 2\ln x + x + 2x^{-1/2} + C \to$ A/

③ $8 - x - x^2 = \frac{1}{2}x + \frac{7}{2} \Rightarrow x = -3$ or $3/2$
$dy/dx = -1-2x$; at $x=-3$, $dy/dx = -1-2(-3) = 5 \to$ D/

④ $A = \pi r^2$; $dA/dr = 2\pi r$; $dA = 2\pi r\,dr$;
$dA = 2\pi(2)(.02) = 2\pi/25 \to$ A/

⑤ $T = \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + y_3)$
$= \frac{\pi/2}{2\cdot 3}\left(1 + 2\cdot\frac{\sqrt{3}}{2} + 2\cdot\frac{1}{2} + 0\right)$
$= \pi(2+\sqrt{3})/12 \to$ A/

⑥ I is false since $g(x)=0$ at $x=0$.
II is false since $g(x)$ is decreasing on $(-2,0)$. III is false since $g''(x) = 3e^x(x^2+4x+2)$ changes sign twice. IV is false since $g(x)$ has a rel max at $x=-2$. $\to$ E/

⑦ $\frac{du}{dx} = \frac{x}{\sqrt{x^2+9}}$, $\frac{dv}{dx} = 9x^2 - 2$;
$du/dv = \frac{du/dx}{dv/dx} = \frac{x}{(9x^2-2)\sqrt{x^2+9}}$
$= \frac{4}{(142)(5)} = \frac{2}{355} \to$ D/

⑧ $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$
$= \lim_{h\to 0}(6x+3h) = 6x$ so $f'(2) = 12$.
The antiderivative of $6x$ is $3x^2 + C$, so $f(x) = 3x^2 + C$ and since $f(1) = 5$; $f(x) = 3x^2 + 2$ and $f(2) = 14$.
$f(2) + f'(2) = 14 + 12 = 26. \to$ D/

⑨ Examine the first few derivatives of $f(x)$.
$f'(x) = \frac{2}{2x+3}$;

⑨ (cont.) $f''(x) = \frac{-4}{(2x+3)^2}$;
$f'''(x) = \frac{16}{(2x+3)^3} \to$ C/

⑩ Applying l'Hopital's Rule:
$\lim_{x\to 0}\frac{\sin 3x}{2x\,g(x)} = \lim_{x\to 0}\frac{3\cos 3x}{2x\,g'(x) + 2g(x)}$
$= \frac{3(1)}{2(0)(4)+2(-1)} = -3/2 \to$ A/

⑪ $\frac{dV}{dt} = k(20{,}000 - V)$; $\int (20{,}000-V)^{-1}dV = k\int dt$;
$-\ln|20{,}000 - V| = kt + c$; $V = 20{,}000 - Ce^{-kt}$.
since $V(0) = 2000$, $V = 20{,}000 - 18{,}000e^{-kt}$.
since $V(1) = 3000$, $V = 20{,}000 - 18{,}000e^{t\ln 17/18}$.
Therefore $V(3) = 20{,}000 - 18{,}000e^{3\ln 17/18}$
$= 20{,}000 - 18{,}000\left(\frac{17}{18}\right)^3 \to$ D/

⑫ $g'(x) = 3\sin^2(3x)$;
$g''(x) = 6\sin(3x)\cos(3x)\cdot 3 = 9\sin(6x)$;
$g'''(x) = 54\cos(6x) \to$ A/

⑬ $A = \frac{1}{2}ab\sin C = \frac{1}{2}s^2\sin\theta$
$\frac{dA}{dt} = \frac{1}{2}s^2\cos\theta\frac{d\theta}{dt} + s\frac{ds}{dt}\sin\theta$.
$= \frac{1}{2}(2\sqrt{3})^2(\sqrt{3}/2)(\pi/90) + (2\sqrt{3})(-\frac{1}{10})(\frac{1}{2})$
$= \frac{12\sqrt{3}\pi}{360} - \frac{\sqrt{3}}{10} = \frac{\sqrt{3}}{30}(\pi - 3) \to$ D/

⑭ Differentiating with respect to x:
$\cos x = 1 + x\sec^2 y\, dy/dx + \tan y$
so $\frac{dy}{dx} = \frac{\cos x - 1 - \tan y}{x\sec^2 y} = \frac{\cos x - \left(\frac{\sin x - x}{x}\right)}{x\sec^2 y}$
$= \frac{x\cos x - \sin x + x}{x^2\sec^2 y} \to$ C/

⑮ $v(t) = 3(t-8)^{1/3} + C_1$; $v(9) = 6 \Rightarrow$ $C_1 = 3$. $s(t) = \frac{9}{4}(t-8)^{4/3} + 3t + C_2$.
Since $v(t)$ changes sign when $t=7$,
total distance $= \left|\int_0^7 v(t)dt\right| + \left|\int_7^8 v(t)dt\right|$
$= \left|\left(\frac{93}{4}+C_2\right) - (36+C_2)\right| + \left|(24+C_2) - \left(\frac{93}{4}+C_2\right)\right|$
$= \frac{51}{4} + \frac{3}{4} = \frac{54}{4} = \frac{27}{2} \to$ B/

⑯ Neither III nor V have derivatives at $x=0$.
$x|x^2| = x^3$ whose 2nd derivative is $6x$.
choices I & IV are both $|x^3|$ whose second derivative is $6|x|$. $\to$ B/

(17) $f_{avg} = \frac{1}{3-0}\left(\int_0^1 x^3 dx + \int_1^3 x^2 dx\right)$
$= \frac{1}{3}\left(\frac{1}{4}x^4\Big]_0^1 + \frac{1}{3}x^3\Big]_1^3\right) = \frac{107}{36}$.
Since $f(c) = \frac{107}{36}$, c must lie in the interval $[1,3]$ because $x^3 \le 1$ if $x \le 1$. Solving $c^2 = \frac{107}{36} \Rightarrow c = \frac{\sqrt{107}}{6}$ only. → C ✓

(18) $g(x) = h(1+f(x)) = (1+f(x))^2$.
$g'(x) = 2(1+f(x)) \cdot f'(x)$
$g'(1) = 2(1+f(1)) \cdot f'(1)$
$1 = 2(1+f(1)) \Rightarrow f(1) = -\frac{1}{2}$ → B ✓

(19) $V_1 = \pi\int_0^4 x^3 dx = \frac{\pi}{4}x^4\Big]_0^4 = 64\pi$.
$V_2 = \pi\int_0^4 (64 - x^3)dx = \pi\left(64x - \frac{1}{4}x^4\right)\Big]_0^4$
$= 192\pi$. $V_1 : V_2 = 64\pi : 192\pi = 1:3$ → B ✓

(20) $f(x)$ is not continuous at $x=0$.
$g(x)$ is not differentiable at $x=2$.
$h(-2) \ne h(0)$ and $K(1) \ne K(3)$ → E ✓

(21) $5\int (x^2+7)^{-2/3} x\,dx = \frac{5/2\,(x^2+7)^{1/3}}{1/3} + C$
$= \frac{15}{2}(x^2+7)^{1/3} + C$ → C ✓

(22) $C_{avg} = \frac{1}{2-0}\int_0^2 \frac{1}{2}t\cos(t^2)dt$
$= \frac{1}{8}\sin(t^2)\Big]_0^2 = \frac{1}{8}\sin(4)$ → E ✓

(23) $f(x)$ must satisfy 2 conditions. First, $f(0)=1$, which all choices meet. Secondly, $f'(x) = (f(x))^2 + 1$, which only IV meets since $\sec^2(x+\frac{\pi}{4}) = \tan^2(x+\frac{\pi}{4}) + 1$ → D ✓

(24) $y' = \frac{x^2-4x-4}{(x^2+4)^2}$; $y'' = \frac{-2x^3+12x^2+24x-16}{(x^2+4)^3}$
$y'' = \frac{-2(x+2)(x^2-8x+4)}{(x^2+4)^3} = 0$ at $x = -2$ or $x = 4 \pm 2\sqrt{3}$
Inflection pts are $(-2, \frac{1}{2})$, $(4+2\sqrt{3}, \frac{1-\sqrt{3}}{8})$ and $(4-2\sqrt{3}, \frac{1+\sqrt{3}}{8})$. The slope of the line joining these 3 collinear pts is $-\frac{1}{16}$. The equation of the line is $y - \frac{1}{2} = -\frac{1}{16}(x+2)$ → B ✓

(25) $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2+k^2} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{n^2}{n^2+k^2}$
$= \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \frac{1}{1+k^2/n^2} = \int_0^1 \frac{dx}{1+x^2}$
$= \arctan x\Big]_0^1 = \arctan 1 = \pi/4$ → C ✓

(26) $f(x) = \int -2x e^{-x^2} dx = e^{-x^2} + C$, through $(0,1) \Rightarrow C = 0$, so $f(x) = e^{-x^2}$.

(26) (cont.) Regard $g(x)$ as $\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}$ $= F'(x)$ where $F(x) = \int_1^x e^{-t^2}dt$. So, $g(x) = e^{-x^2}$. Using the quotient rule,
$h(x) = \frac{d}{dx}\left(\frac{-e^{-x^2}}{2x}\right) = e^{-x^2}\left(1 + \frac{1}{2x^2}\right)$. A ✓

(27) Area $= \int_b^1 3x^2 dx + \int_1^{b+2} (4-x)dx$
$= x^3\Big]_b^1 + \left(4x - \frac{1}{2}x^2\right)\Big]_1^{b+2} = -b^3 - \frac{1}{2}b^2 + 2b - \frac{9}{2}$.
$A' = -3b^2 - b + 2 = (2-3b)(b+1) = 0$ at $b = -1$ or $2/3$. Choose $b = 2/3$. B ✓

(28) The rate at which the disease is spreading is $f'(t) = 3Pe^{-t}(1+3e^{-t})^{-2}$.
This rate will be maximized when $\frac{d}{dt}f'(t) = f''(t) = 0$. $f''(t) = 3Pe^{-t}$ .....
$(-2)(1+3e^{-t})^{-3}(-3e^{-t}) + (1+3e^{-t})^{-2}(-3Pe^{-t})$
$= 3Pe^{-t}(3e^{-t}-1)(1+3e^{-t})^{-3} = 0$ at $t = \ln 3$.
$\frac{f(\ln 3)}{P} = \frac{P}{(1+3(\frac{1}{3}))P} = 1/2$ → C ✓

(29) $y' = -10(5x+2)^{-3}$; $m_{norm} = \frac{1}{10}(5x+2)^3$
$= \frac{25}{2} \Rightarrow x = \frac{3}{5}$, $y = \frac{1}{25}$;
$y - \frac{1}{25} = \frac{25}{2}\left(x - \frac{3}{5}\right)$ or $625x - 50y = 373$
→ A ✓

(30) Using u-substitution: let $u = \sqrt{2x-1}$
$\Rightarrow x = \frac{1}{2}(u^2+1) \Rightarrow dx = u\,du$
$\int_{1/2}^1 3x\sqrt{2x-1}\,dx = \frac{3}{2}\int_0^1 (u^4+u^2)du$
$= \left(\frac{3}{10}u^5 + \frac{1}{2}u^3\right)\Big]_0^1 = \left(\frac{3}{10}+\frac{1}{2}\right) - 0$
$= \frac{8}{10} = \frac{4}{5}$ → D ✓