Complex Numbers (Alpha) — Solutions

FAMAT State Convention 2004

1. Since $|z|^2 = [\Re(z)]^2 + [\Im(z)]^2$, we conclude that $\Im(z) = \pm\sqrt{|z|^2 - [\Re(z)]^2} = \pm\sqrt{9^2 - 4^2} = \pm\sqrt{65}$. $\boxed{\textbf{A}}$

2. $\dfrac{m}{n} = \left|\dfrac{6+8i}{-20+21i}\right| = \dfrac{|6+8i|}{|-20+21i|} = \dfrac{10}{29}$. Therefore, $m + n = 10 + 29 = 39$. $\boxed{\textbf{B}}$

3. Since $2004 \equiv 0 \pmod 4$, we have $i^{2004} = i^0 = 1$. $\boxed{\textbf{C}}$

4. $\dfrac{(2+3i)(4+5i)}{i} = (2+3i)(4+5i)(-i) = (-7+22i)(-i) = 22 + 7i$. $\boxed{\textbf{D}}$

5. The fifth roots of 1 are precisely the roots of $z^5 - 1 = 0$. Since this equation has 0 as the sum of the roots, taking 1 away from that leaves -1 as the sum of the remaining roots. $\boxed{\textbf{A}}$

6. For n a positive integer, there are always n distinct nth roots of any given non-zero complex number. $\boxed{\textbf{E}}$

7. The portion of the angle in the first quadrant is $\arctan\dfrac{1}{3} \approx 18.4^\circ$, the portion in the second quadrant is 90°, and the portion in the third quadrant is again $\arctan\dfrac{1}{3} \approx 18.4^\circ$, so the entire angle is approximately $18.4^\circ + 90^\circ + 18.4^\circ \approx 127^\circ$. $\boxed{\textbf{B}}$

8. Since $(-i)^6 = -1 \neq 1$, it is not a sixth root of 1. $\boxed{\textbf{B}}$

9. The resultant vector is approximately $0.04328 + 1.14267i$, which has a length of about 1.14. Thus, $r \approx \pm 1.14$. $\boxed{\textbf{B}}$

10. All of the given numbers are complex numbers. $\boxed{\textbf{E}}$

11. $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{2004} = (\operatorname{cis} 45^\circ)^{2004} = \operatorname{cis}(2004 \times 45^\circ) = \operatorname{cis} 180^\circ = -1$. $\boxed{\textbf{B}}$

12. If $z = x + iy$, then $z^2 + \overline{z}^2 = x^2 - y^2 + 2ixy + x^2 - y^2 - 2ixy = 2x^2 - 2y^2 = 2$, which is a hyperbola. $\boxed{\textbf{D}}$

13. $\arg(7-6i) = \tan^{-1}\left(-\dfrac{6}{7}\right) \approx -0.70863 + k\pi$, where k is an integer. Taking $k = 1$ yields approximately 2.43. $\boxed{\textbf{C}}$

14. $(\cos 30^\circ + i\sin 30^\circ)(e^{11\pi i/6}) = (\cos 30^\circ + i\sin 30^\circ)(\cos 30^\circ - i\sin 30^\circ) = \cos^2 30^\circ + \sin^2 30^\circ = 1$. $\boxed{\textbf{D}}$

15. Plot the points and you will see that there are only three possible vertices to complete the parallelogram. The point $8 - 2i$ is not one of them. $\boxed{\textbf{C}}$

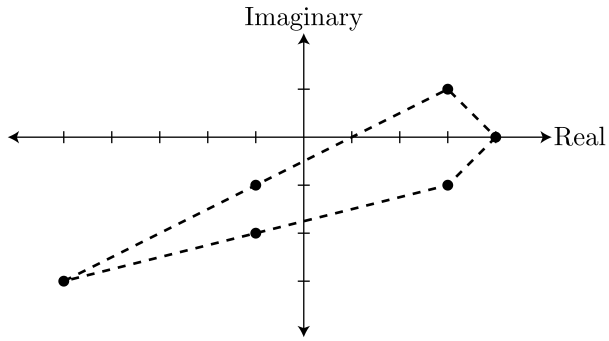
16. $\dfrac{1}{i} + \dfrac{1}{i^2} + \dfrac{1}{i^3} + \dfrac{1}{i^4} = (-i) + (-1) + i + 1 = 0$. $\boxed{\textbf{E}}$

17. Since the sum of the distances between a point on the graph (z) and two fixed points (the foci at 1 and $-i$) is given to be a constant (2), and since that constant is greater than the distance between the two foci, we have an ellipse. $\boxed{\textbf{D}}$

18. If $z = a + bi$ and both a and b are real, then $z\overline{z} = a^2 + b^2$, which here is $(-2)^2 + 7^2 = 53$. $\boxed{\textbf{E}}$

19. Let $e^{i\theta} = \operatorname{cis}\theta = -i$. Then $\cos\theta = 0$ and $\sin\theta = -1$, so $\theta = \dfrac{3\pi}{2} + 2k\pi$, and $-i = e^{\left(\frac{3\pi}{2} + 2k\pi\right)i}$. Thus, $\ln(-i) = \dfrac{3\pi i}{2} + 2k\pi i$, and taking $k = 0$ we get $\operatorname{Ln}(-i) = \dfrac{3\pi i}{2}$. $\boxed{\textbf{C}}$

20. After plotting the points, we have the following figure.



This is a quadrilateral. $\boxed{\textbf{C}}$

21. $\sqrt{-5} \times \sqrt{-28} \times \sqrt{-35} = i\sqrt{5} \times i\sqrt{28} \times i\sqrt{35} = i^3 \times \sqrt{4900} = (-i) \times 70 = -70i.$ $\boxed{\textbf{A}}$

22. $m + n \div p = (1+2i) + (2-3i) \div (-3+4i) = (1+2i) + \left(-\frac{18}{25} + \frac{i}{25}\right) = \frac{7}{25} + \frac{51i}{25}.$ $\boxed{\textbf{B}}$

23. By the rule of signs, the number of negative real roots can be at most equal to the number of sign changes in $p(-z) = 573z^4 + 892z^3 + 2z^2 - 573z - 89$. Since there is only one sign change, and since the real coefficients imply that the parity of the number of negative real roots equals the parity of the number of sign changes in $p(-z)$, there must be exactly one negative real root. $\boxed{\textbf{B}}$

24. Since $f(z) = z^2 + 4iz - 4 = (z+2i)^2$, we get $f(g(3+2i)) = f(3-2i) = (3-2i+2i)^2 = 9.$ $\boxed{\textbf{E}}$

25. I is true. II is false; as a counterexample, the number $1+i$ is not in the set $\mathbb{R} \cup \{xi : x \in \mathbb{R} \text{ and } i = \sqrt{-1}\}$ (in fact, that set only contains the two axes of the complex plane). III is true; each succeeding set of numbers is built up from the preceding one. $\boxed{\textbf{C}}$

26. $\overline{r \operatorname{cis} \theta} = \overline{r}\overline{\operatorname{cis} \theta} = r(\overline{\cos\theta + i\sin\theta}) = r(\cos\theta - i\sin\theta) = r[\cos(-\theta) + i\sin(-\theta)] = r\operatorname{cis}(-\theta).$ $\boxed{\textbf{D}}$

27. Since $|1-i| = \sqrt{2}$, the only possible value for n is $n = \log_{\sqrt{2}} 4096 = 2\log_2 4096 = 2(12) = 24$. Indeed, $(1-i)^{24} = \left(\sqrt{2}\operatorname{cis}(-45^\circ)\right)^{24} = 4096\operatorname{cis}(24(-45^\circ)) = 4096\operatorname{cis}0^\circ = 4096.$ $\boxed{\textbf{D}}$

28. $z^3 = [(1-i)^4]^3 = (1-i)^{12} = [\sqrt{2}\operatorname{cis}(-45^\circ)]^{12} = (\sqrt{2})^{12}\operatorname{cis}[12 \times (-45^\circ)] = 64\operatorname{cis}(-540^\circ) = 64(-1) = -64.$ $\boxed{\textbf{A}}$

29. The product of any two imaginary numbers is always real (and hence complex), never imaginary, and can be either not negative ($-i \times i = 1$) or not positive ($i \times i = -1$). Thus, none of the listed choices is correct. $\boxed{\textbf{E}}$

30. The points form an isosceles triangle with a base length of 8 and a height of 5. The area is then given by $\frac{1}{2} \mathit{base} \times \mathit{height} = \frac{1}{2} \times 8 \times 5 = 20.$ $\boxed{\textbf{B}}$