ALPHA LOGS AND EXPONENTS SOLUTIONS
FAMAT State Convention 2004

1. Take the natural log of both sides:
   $\ln\left(e^{2x}\right) = \ln(3) \quad 2x = \ln(3)$
   $x = \frac{\ln(3)}{2} \approx 0.549$
   Answer: 0.549; D
2. Rewrite it with a change of base and multiply by -1 on the top and bottom:
   $\log_a b = \frac{\log b}{\log a} = \frac{-\log b}{-\log a} = \frac{\log\left(b^{-1}\right)}{\log\left(a^{-1}\right)} = \log_{\frac{1}{a}} \frac{1}{b}$
   Answer: $\log_{\frac{1}{a}} \frac{1}{b}$; B
3. Plug the points into the function and solve the system for a and b.
   $f(0) = 2: \ ab^0 = 2 \ \text{so} \ a = 2$
   $f(1) = 6: \ ab^1 = 2b = 6 \ \text{so} \ b = 3$
   Now find f(5).
   $f(5) = 2 * 3^5 = 486$
   Answer: 486; B
4. Answer: 1.21; C
5. $\log_{10}\left(2^{1000}\right) = 1000\log_{10}(2)$
   $\approx 1000 * .301 = 301$
   Answer: 301; C
6. Take the log-base 10 of the number.
   $\log_{10}\left(7^{89}\right) = 89 * \log_{10}(7) \approx 75.2137$
   This shows that 75 digits are insufficient to represent the number but 76 work.
   Answer: 76; D
7. Let $y = e^x$ so the equation becomes $2y^2 - 4y + 1 = 0$. The solutions are $y = \frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}$. We change these to the corresponding x values by taking the natural logarithm of the y values: $x = \ln\left(\frac{2+\sqrt{2}}{2}\right), \ln\left(\frac{2-\sqrt{2}}{2}\right)$.
   Use the properties of logarithms to find the sum.
   $\ln\left(\frac{2+\sqrt{2}}{2}\right) + \ln\left(\frac{2-\sqrt{2}}{2}\right) = \ln\left(\left(\frac{2+\sqrt{2}}{2}\right)\left(\frac{2-\sqrt{2}}{2}\right)\right) = \ln\left(\frac{4-2}{4}\right) = \ln\left(\frac{1}{2}\right)$
   Answer: $\ln\left(\frac{1}{2}\right)$; A
8. Repeatedly apply the recursive relation to find $f\left(2^n\right)$ for n a positive integer:
   $f\left(2^n\right) = 2^n * f\left(2^{n-1}\right) = 2^n * 2^{n-1} * f\left(2^{n-2}\right) =$
   $2^n * 2^{n-1} * \cdots 2^2 * 2 * f(1) = 2^{n+(n-1)+\cdots+2+1}$
   $f\left(2^n\right) = 2^{\frac{n(n+1)}{2}}$
   $f(256) = f\left(2^8\right) = 2^{\frac{8*9}{2}} = 2^{36}$
   The log-base 2 of this number is 36.
   Answer: 36; D
9. Expand using Euler's Formula:
   $e^{ix} = \cos(x) + i\sin(x)$ gives
   $e^{5ix} + e^{-5ix} = \cos(5x) + i\sin(5x) + \cos(-5x) + i\sin(-5x)$
   Using the even/odd properties of cos and sin gives $2\cos(5x)$.
   Answer: $2\cos(5x)$; E
10. Convert to polar form and then use DeMoivre's Theorem.
    $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{10} = \left(\text{cis}\left(-\frac{\pi}{3}\right)\right)^{10} = \text{cis}\left(-\frac{10\pi}{3}\right) = \text{cis}\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
    Answer: $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$; C
11. $16x = (2x)^{\log_2 x} \qquad 16x = 2^{\log_2 x} x^{\log_2 x}$
    $16x = x * x^{\log_2 x} \qquad 16 = x^{\log_2 x}$
    $\log_x 16 = \log_2 x \qquad \frac{\log_2 16}{\log_2 x} = \log_2 x$
    $4 = \left(\log_2 x\right)^2 \qquad \log_2 x = \pm 2$
    $x = 4, \frac{1}{4}$
    Answer: 1; A
12. $\log_{10} 1 + \log_{10} 2 + ... + \log_{10} 10 =$
    $\log_{10}(10!) = \log_{10}(3628800) \approx 6.56$
    Answer: 7; D
13. Take out all factors raised to an even power from the radical.
    $35000 = 2^3 * 5^4 * 7$

ALPHA LOGS AND EXPONENTS SOLUTIONS
FAMAT State Convention 2004

$\sqrt{35000} = 50\sqrt{14}$

$50 + 14 = 64$

Answer: 64; B

14. $\frac{1}{a} = \log_5 12 \qquad 2b = \log_5 3$

$\frac{1}{a} - 2b = \log_5 12 - \log_5 3 = \log_5 4$

$\frac{1}{2}\left(\frac{1}{a} - 2b\right) = \frac{1-2ab}{2a} = \log_5 2$

Answer: 2a; D

15. You multiply the exponents in this case: $\left(2^3\right)^2 = 2^6$.

Answer: 2^6; C

16. Take the log-base 2 of the number.

$\log_2(1576) \approx 10.622$

The integer part of the logarithm gives the number of bits in the number.

Answer: 10; C

17. Using binomial expansion the term we are interested in is

$\binom{10}{7}(2x)^7(-3)^3 = -120*128*27x^7 = -414720x^7$

Answer: -414720; E

18. Since x is a complex number, use DeMoivre's Theorem. Raising x to the nth power multiplies the angle of the complex number by n. We want all of the x^n to be 1 so the angle must be a multiple of 2π. Any multiple of 6 will work for n since the 6 will cancel out with the 3 in the denominator of the angles and leave a factor of 2 to be multiplied by π. Alternatively notice that the set given contains all of the 6th roots of unity and thus are roots of $x^6 = 1$.

Answer: 6; B

19. I. $\log_5 2 = \frac{1}{\log_2 5}$ and I does not work

II. $\log_8 5 = \log_{\left(2^3\right)} 5 = \frac{1}{3}\log_2 5$ and II works

III. $\log_{0.5} 0.2 = \log_{\left(\frac{1}{0.5}\right)}\left(\frac{1}{0.2}\right) = \log_2 5$ and III works.

IV. $\log_2 25 = \log_2\left(5^2\right) = 2\log_2 5$ and IV works.

Answer: 3; C

20. Using the binomial theorem we have $(x+2)^3 = x^3 + 6x^2 + 12x + 8$. Thus the logarithm is 3.

Answer: 3; C

21. $2^{\left(x^2\right)} = 3^{(x+2)}$

$\log_2\left(3^{(x+2)}\right) = x^2$

$(x+2)\log_2(3) = x^2$

$x^2 - \log_2(3)x - 2\log_2(3) = 0$

The sum of the solutions is

$-\frac{b}{a} = \log_2(3)$.

Answer: $\log_2(3)$; C

22. If x is the number of years and y is the diameter in mm, exponential decay means $y = a * b^x$. The original diameter of 10 mm gives $a = 10$. Since the diameter is halved in one year $b = \frac{1}{2}$. We must find x when y is 1. The equation becomes $1 = 10\left(\frac{1}{2}\right)^x$.

$2^x = 10 \quad x = \log_2 10$ Since 1 year has already passed, the number of additional years is $\log_2 10 - 1$.

Answer: $\log_2 10 - 1$; D

23. Work backwards finding the domain of h(x), then h(g(x)), then h(g(f(x))).

Domain of h(x) is $x - 1 > 0$ or $x > 1$.

Domain of h(g(x)) is $\sqrt{\frac{x}{2}} > 1$ or $\frac{x}{2} > 1$ or $x > 2$.

ALPHA LOGS AND EXPONENTS SOLUTIONS
FAMAT State Convention 2004

Domain of h(g(f(x))) is $10^x > 2$

$x > \log_{10} 2$.

Answer: $x > \log_{10} 2$; E

24. Use the double angle identity to rewrite $\cos^2(x)$.

$$\cos^4(x) = \left(\frac{\cos(2x)+1}{2}\right)^2 = \frac{\cos^2(2x) + 2\cos(2x) + 1}{4}$$

Apply the identity again.

$$= \frac{\frac{\cos(4x)+1}{2} + 2\cos(2x) + 1}{4} = \frac{\cos(4x)}{8} + \frac{\cos(2x)}{2} + \frac{3}{8}$$

$$a + b + c = \frac{1}{8} + \frac{1}{2} + \frac{3}{8} = 1$$

Answer: 1; A

25. I. True.

II. False if y is even and x is negative. For example, $\sqrt{x^2} = |x|$.

III. False, $p^{\log_p q} = q$.

Answer: 1; B

26. Convert to polar form.

$$(3+4i) = 5\text{cis}\left(\tan^{-1}\frac{4}{3}\right) = 5\text{cis}\,(.9273)$$

Use DeMoivre's Theorem:

$$(5\text{cis}\,(.9273))^8 = 5^8\,\text{cis}\,(8 * .9273) = 5^8\,\text{cis}\,(1.135)$$

$\frac{b}{a}$ is the tangent of the angle of the complex number. Therefore,

$$\frac{b}{a} = \tan(1.135) = 2.148$$

Answer: 2.15; D

27. By inspection it is clear that x=2 is a solution to the equation. Thinking about the graphs of the functions determines that they must intersect for exactly one negative value of x (x=-.7667). We know the exponential enters the inside of the parabola on the left and then cuts through the parabola at x=2. Since exponential functions grow faster than polynomial functions, there must also be a point where the exponential function overtakes the parabola. This happens at x=4.

Answer: 3; D

28. $x = pe^{rt}$ $\quad 2p = pe^{.1t}$

$2 = e^{.1t}$ $\quad t = \frac{\ln 2}{.1}$

Answer: 6.93; A

29. Rewrite the equation using $2^2 = 4$.

$4^{(x+1)} = 2^{(x^2)}$ $\quad 2^{(2x+2)} = 2^{(x^2)}$

Now equate the exponents.

$2x + 2 = x^2$ $\quad x^2 - 2x - 2 = 0$ The sum of the solutions is $-\frac{b}{a} = 2$.

Answer: 2; D

30. $x - [x]$ is the fractional part of x. The fractional part of $\frac{97^{2004}}{5}$ is determined by the remainder. Look at the remainders using small exponents.

The remainder of 97 is 2.

The remainder of 97^2 is 2^2 or 4.

The remainder of 97^3 is $2^3 - 5$ or 3.

The remainder of 97^4 is $2^4 - 15$ or 1.

Therefore, the remainder of 97^{2004} is 1 because 2004 is a multiple of 4.

Therefore, the fractional part of $\frac{97^{2004}}{5}$ is $\frac{1}{5} = 0.2$.

Answer: 0.2; B