Solutions to Calculus Individual – FAMAT State Convention 2004

D 1. Use L'Hopital's Rule twice to get

$$=\lim_{x\to 0}\frac{\cos(2x)+x\cos x+\sin x-e}{96x\cos x+96\sin x}$$

$$=\lim_{x\to 0}\frac{-2\sin(2x)+2\cos x-x\sin x-e^x}{192\cos x-96x\sin x}=\frac{1}{192}$$

D 2. I. True by Intermediate Value Theorem
II. Not necessarily true because f is not necessarily differentiable on (2,5).
III. True by Intermediate Value Theorem

A. 3. Solve $600-16t^2=0$ $\quad f'(t)=-32t$

$$t=\frac{5\sqrt{6}}{2} \qquad f'\left(\frac{5\sqrt{6}}{2}\right)=-80\sqrt{6}$$

B 4.

$$\frac{\sqrt{7x-10}-5}{10(x-5)}\cdot\frac{\sqrt{7x-10}+5}{\sqrt{7x-10}+5}=\frac{7x-10-25}{10(x-5)\cdot\sqrt{7x-10}+5}$$

$$=\frac{7x-35}{10(x-5)\sqrt{7x-10}+5}=\frac{7(x-5)}{10(x-5)\sqrt{7x-10}+5}$$

$$\lim_{x\to 5^+}\frac{7}{10\left(\sqrt{7x-10}+5\right)}=\frac{7}{100}$$

B 5. $y'=3x^2+4x-4=11$

$3x^2+4x-15=0$

$(3x-5)(x+3)=0$

$x=\frac{5}{3}$ or $x=-3$

C 6.

$h'(x)=f'(g(x))\cdot g'(x)$

$h''(x)=f'(g(x))\cdot g''(x)+g'(x)\cdot f''(g(x))\cdot g'(x)$

$h''(2)=f'(2)\cdot g''(2)+4f''(2)$

$=-g''(2)+4f''(2)=-f''(2)+4f''(2)=3f''(2)$

A 7. $y=\int\sqrt{2x+1}\,dx=\frac{1}{2}\cdot\frac{2}{3}(2x+1)^{\frac{3}{2}}+C$

$$\frac{y(4)-y(0)}{4}=\frac{\frac{1}{3}(27-1)}{4}=\frac{26}{12}=\frac{13}{6}$$

D 8. $V=\frac{4}{3}\pi r^3 \quad dV=4\pi r^2dr$

$dV=4\pi(2)^2(.01) \quad dV=.16\pi\approx 0.503$

A 9. $\frac{dw}{dz}=\frac{1}{27}+\frac{1}{z^2} \quad \frac{dz}{dx}=3\left(x^2+2\right)^2\cdot 2x$

$\frac{dz}{dx}\Big|_{x=1}=54 \quad \frac{dx}{dz}\Big|_{z=27}=\frac{1}{27}+\frac{1}{27^2}=\frac{28}{27^2}$

$\frac{28}{27^2}\cdot 54=\frac{56}{27}$

C 10. $\frac{d^2y}{dx^2}=\frac{4y}{2\sqrt{2y^2-1}}\cdot\sqrt{2y^2-1}=2y$

$\frac{d^2y}{dx^2}\Big|_{y=3}=2(3)=6$

B 11. $f'(x)=e^x-e^{-x}-2\sin x$

$f'(0)=0, f'(-.1)<0, f'(.1)>0$

f has a local minimum at $x = 0$.

C 12. $y'=-2\sin(2x)+2\cos(2x)=0$

$1=\tan(2x)$

$2x=\frac{\pi}{4}+k\pi;\ x=\frac{\pi}{8}+\frac{k}{2}\pi$

The function has a period of π.

$f\left(\frac{\pi}{8}\right)=\sqrt{2}$ and $f\left(\frac{5\pi}{8}\right)=-\sqrt{2}$

The maximum value is $\sqrt{2}$,

B 13. I. True because f is strictly decreasing.
II. True because f' is decreasing.
III. Not necessarily true.
IV. True by the mean value theorem applied to f'.

A 14. This series is equal to

$$\int_o^\pi \sin x\,dx=-\cos x\Big]_0^\pi=-\cos\pi--\cos 0=2$$

E 15. $g'(x)=\frac{x}{e^x+1};\ g''(x)=\frac{e^x+1-xe^x}{\left(e^x+1\right)^2}$

$g''(0)=\frac{1}{2}$

D 16. $u = x^2 \quad u(1) = 1 \quad u(2) = 4 \quad \frac{1}{2}\int_1^4 e^u\,du$

C 17. $\int_0^{\frac{\pi}{4}} (\cos x - \sin x)\,dx + \int_{\frac{\pi}{4}}^{\pi} (\sin x - \cos x)\,dx =$

$(\sin x + \cos x)\Big]_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big]_{\frac{\pi}{4}}^{\pi} =$

$\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - 1 + 1 - 0 - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = 2\sqrt{2}$

A 18. $\frac{dy}{dx}\Big|_{(2,0)} = \frac{2xy^3 - 6x}{4 - 3x^2y^2}\Big|_{(2,0)} = \frac{-12}{4} = -3$

$dy = -3dx \quad dy = -3(-.03) = .09$

A 19. $v'(t) = a(t) = \frac{2x\cos x - \sin x}{2x^{\frac{3}{2}}}$

$a(3.5) < 0$ and $v(3.5) < 0$

means speed is increasing.

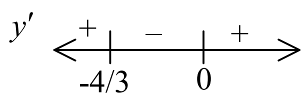
A 20. When $x < 0$ slopes are negative and when $x > 0$ slopes are positive. When $x = 0$, slopes are 0.

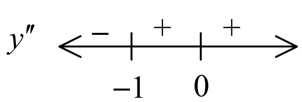
B 21. $y' = 20x^3 + 15x^4 = 5x^3(4 + 3x) = 0$

$x = 0$ or $x = -4/3$

$y'' = 60x^2 + 60x^3 = 60x^2(1 + x) = 0$

$x = 0$ or $x = -1$





The graph has a local max, a local min, and an inflection point.

D 22. $\int_0^3 \sqrt{x+1}\,dx + \int_3^4 (2x - 4)\,dx =$

$\frac{2}{3}(x+1)^{\frac{3}{2}}\Big]_0^3 + (x^2 - 4x)\Big]_3^4$

$= \frac{16}{3} - \frac{2}{3} + 0 + 3 = \frac{23}{3}$

E 23. $6x + 2 = 4 \quad 6x = 2 \quad x = \frac{1}{3}$

$4\left(\frac{1}{3}\right) - y = 3 \quad y = -\frac{5}{3}$

$-\frac{5}{3} = 3\left(\frac{1}{3}\right)^2 + \frac{2}{3} + k \quad k = -\frac{8}{3}$

B 24.

I. $h(2) = f(g(2)) \approx f(0) \approx 5$ NO

II. $h'(3) = f'(g(3)) \cdot g'(3) \approx f'(.25) \cdot + = - \cdot + = -$ NO

III. $h'(1) = f'(g(1)) \cdot g'(1) = f'(1) \cdot - = 0$ YES

IV. $h''(x) = f'(g(x)) \cdot g''(x) + (g'(x))^2 \cdot f''(g(x))$

$h''(4) = f'(g(4)) \cdot g''(4) + (g'(4))^2 \cdot f''(g(4))$

$\approx 0 + 1 \cdot f''(1) =$ a positive number

since f is concave up at 1. YES

A 25. $r = \frac{3}{4}h \quad V = \frac{3\pi}{16}h^3 \quad \frac{dV}{dt} = \frac{9\pi}{16}h^2\frac{dh}{dt}$

$6\pi = \frac{9\pi}{16} \cdot 16 \cdot \frac{dh}{dt} \quad \frac{dh}{dt} = \frac{2}{3} \quad A = \pi r^2$

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt} \quad \frac{dA}{dt} = 2\pi \cdot 3 \cdot \frac{1}{2} \quad \frac{dA}{dt} = 3\pi\frac{\text{ft}^2}{\text{min}}$

D 26.

$u = x^2 \quad \int_0^2 \left(xf'(x^2) + 1\right)dx = \frac{1}{2}\int_0^4 f'(u)\,du + \int_0^2 1\,dx$

$= \frac{1}{2}f(u)\Big]_0^4 + 2 = \frac{1}{2}\left(f(4) - f(0)\right) + 2 = \frac{1}{2} \cdot 12 + 2 = 8$

B 27. $f'(x) = \frac{2}{2x+1} \quad f''(x) = \frac{-4}{(2x+1)^2} \quad f(0) = 0;$

$f'(0) = 2; \; f''(0) = -4 \quad 2 + -4 = -2$

C 28. $f'(x) = e^{-kx}(-kx + 1) \quad f''(x) = -ke^{-kx}(2 - kx)$

$f''(x) = 0$ when $x = \frac{2}{k} \quad f''(0) < 0 \quad f''\left(\frac{3}{k}\right) > 0$

f is concave down for $x < \frac{2}{k}$.

B 29. The zeros of f are 1 and –7/2.

$f''(x) = 12x + 6 \quad f''(1) = 18 \quad f''\left(-\frac{7}{2}\right) = -36$

D 30.

$g(-1) = -2 \quad f(-2) = -9 \quad m = f'(-2) \cdot g'(-1) = 19$

$y - (-9) = 19(x + 1)$ or $19x - y = -10$