Sequences and Series—SOLUTIONS   State Convention 2004

1. C—If \( |r| > 1 \), then it diverges (b/c its unbounded) and if \( |r| < 1 \) then the limit = 0, hence converges. If \( r = 1 \), converges to 0 and if \( r = -1 \), it diverges by oscillation. So \(-1 < r \leq 1\).

2. B—\( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \ldots + \frac{1}{n(n+1)} \). So, \( S_n = 1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4} - \ldots - \frac{1}{n} + \frac{1}{n+1} = 1 - \frac{1}{n+1} \). And, \( \lim_{n \to \infty} S_n = 1 \).

3. D—The limit of the ratio for the series \( \sum \frac{1}{n^{1/2}} \) is 1, so this test fails.

4. D—\( S = \frac{a}{1-r} = \frac{2/3}{1-1/3} = 2 \).

5. C—\( \lim_{n \to \infty} (n+1)(x-3) = \infty \) (diverges), unless \( x = 3 \).

6. D—The series is \( \cos \frac{\pi}{4} - (x-\frac{\pi}{4}) \sin \frac{\pi}{4} - \frac{(x-\frac{\pi}{4})^2}{2!} \cos \frac{\pi}{4} + \frac{(x-\frac{\pi}{4})^3}{3!} \sin \frac{\pi}{4} - \ldots \) And the coefficient of the required term is \( \frac{\sqrt{2}}{12} \).

7. A—Using the series for \( e^x \) and letting \( x = -0.1 \) gives the answer.

8. A—\( f(x) = x \ln x; \ f'(x) = 1 + \ln x; \ f''(x) = \frac{1}{x}; \ f'''(x) = -\frac{1}{x^2}; \ f^4(x) = \frac{2}{x^3}; \ f^5(x) = -\frac{3 \cdot 2}{x^4} \) and \( f^5(1) = -3 \cdot 2 \).

So, the coefficient of \( (x-1)^3 \) is \( -\frac{3 \cdot 2}{5!} = -\frac{1}{20} \).

9. C—Using the Ratio Test, we get \( \lim_{n \to \infty} \left| \frac{x(1 + \frac{1}{n})^n}{\frac{x}{2} e} \right| = \left| \frac{x}{2} \right| < 1 \), that is, when \( |x| < \frac{2}{e} \). So, the radius of convergence is \( \frac{2}{e} \).

10. D—Since the ratio \( r < 1 \), the sum of the series is \( \frac{a}{1-r} = \frac{\pi^3}{3} \cdot \frac{1}{1-\frac{\pi^3}{3}} \), which simplifies to get the answer.

11. A—The error of this given series is less in absolute value than the first term dropped, that is, less than \( \left| \frac{(-1)^{300}}{900-1} \right| \approx 0.0011 \). The closest answer to this is A.

12. E—Using the Ratio Test, we get \( \lim_{n \to \infty} \frac{1}{n+1} |x-1| \). This equals 0 if \( x \neq 1 \Rightarrow \) convergent everywhere. It also converges if \( x = 1 \).

13. C—The power series for \( \ln(1-x) \), if \( x < 1 \), is \(-x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots \) and the coefficient of the \( x^3 \) term is \( -\frac{1}{3} \).

14. A—Rewrite the given limit as: \( \lim_{n \to \infty} \left[ \left( \frac{1}{n} \right)^{1/2} + \left( \frac{2}{n} \right)^{1/2} + \left( \frac{3}{n} \right)^{1/2} + \ldots + \left( \frac{n}{n} \right)^{1/2} \right] \). Note that the interval in this case is 0 to 1 and \( \Delta x = \frac{1}{n}, \ x_k = \frac{k}{n}, \) and \( f(x_k) = (x_k)^{1/2} \). This gives the integral.
15. B—Differentiating, the new series is 
\[ 1 + \frac{x-1}{2} + \frac{(x-1)^2}{3} + \frac{(x-1)^3}{4} + \ldots \] Using Ratio Test gives the answer.

16. B—The limit approaches 0 as n gets large (denominator gets huge).

17. D—Using the p-Series test, where p > 1, as in D, will show convergence.

18. B—Using the Integral test for B, shows divergence.

19. D—Using the Limit Comparison Test shows convergence for this series.

20. A—Since the given series is a convergent geometric series with a = 2 and r = 0.2, we have
\[ R_3 = \frac{ar^3}{1-r} = \frac{2(1/5)^3}{1-1/5} = 0.02 \]

21. D—We want \( \frac{2(1/5)^n}{1-1/5} < 0.0002 \). Solving, we get \( n > 5.86 \). so 6 terms are required for stated accuracy.

22. D—Rationalizing the denominators, we have \( (\sqrt{2} - \sqrt{1}) + (\sqrt{3} - \sqrt{2}) + (\sqrt{4} - \sqrt{3}) + \ldots + (\sqrt{25} - \sqrt{24}) = 4 \).

23. C—Subtract 2 from the beginning index of summation means add 2 to the formula that is summed.

24. B—We have \( t_{n+1} + t_2 + \ldots + t_{n-1} + t_n = n^2 t_n \Rightarrow (n-1)^2 t_{n-1} = (n^2-1)t_n \Rightarrow t_n = \frac{n-1}{n+1} t_{n-1} \). So the sum is \( \sum_{i=1}^{50} \frac{1}{50+5i} = \frac{50}{51} \).

25. B—Write the series as \( 1 + 3x + 5x^2 + 7x^3 + \ldots \) which then can be written as
\[ [1 + x + x^2 + x^3 + \ldots] + [2(x + x^2 + x^3 + \ldots)] + [2(x^2 + x^3 + \ldots)] + \ldots = \left[ \frac{1}{1-x} \right] + \left[ \frac{2x}{1-x} \right] + \left[ \frac{2x^2}{1-x} \right] + \ldots = \frac{1+x}{(1-x)^2} \]
Letting \( x = \frac{1}{3} \) gives the answer 3.

26. B—Plugging in, one can readily see that everything cancels but \( 2(n+1) - 1 = 2n+1 \).

27. B—Let \( S = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots \) and \( \frac{1}{5} S = \frac{1}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \ldots \) Subtracting these quantities gives
\[ \frac{4}{5} S = \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots \text{ or (using the infinite geom. sum formula),} \]
\[ \frac{4}{5} S = \frac{1/5}{1-1/5} \Rightarrow S = \frac{5}{16} \]

28. C—Rewrite the series as \( 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} + \ldots = 1 - \frac{1}{201} = \frac{200}{201} \).

29. D—Using the geometric sum formula, we get \( S = (1+i)^2 - 1 \) and \( (1+i)^2 = 2i \), so
\[ (1+i)^2 = (2i)^1 \Rightarrow S = -1 - i \cdot 2^{11} = -1 - 2048i \]

30. D—Using the formula given, when \( n = 1 \), \( x_2 = 3 \) and when \( n = 2 \), \( x_3 = 5 \).