Discrete Mathematics
FAMAT State Convention 2004

For all questions, E. NOTA means none of the above answers is correct.

1. Paul would like to determine the relative salaries of three coworkers using two facts. First, he knows that if Noah is not the highest paid of the three, then Graham is. Second, he knows that if Graham is not the lowest paid, then John is the most. What is the order of decreasing relative salary?

    A. Noah, Graham, John
    B. John, Noah, Graham
    C. Noah, John, Graham
    D. Cannot be determined
    E. NOTA

2. The logic function, **F**, is a function of three inputs, **A**, **B**, and **C**. **F** takes on the values shown in the table for the given values of **A**, **B**, and **C**. (Note: $\overline{A}$ denotes the complement of A, A+B denotes A OR B, $A \cdot B$ denotes A AND B.)

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

Which of the following is an expression for **F**?

A. $F = \left(\overline{A} \cdot B \cdot C\right) + \left(A \cdot B \cdot C\right)$

B. $F = \overline{\left(A \cdot B \cdot C\right)} + \left(A \cdot B \cdot C\right)$

C. $F = \overline{\left(\overline{A} \cdot B \cdot C\right) + \left(A \cdot B \cdot C\right)}$

D. $F = \left(A \cdot B \cdot C\right)$

E. NOTA

3. On a sunny Florida afternoon, Alexander proclaims, "I am the best rapper in Gainesville and I also love to eat chicken wings." What is the negation of his statement?

    A. "I am not the best rapper in Gainesville and I do not love eating chicken wings."
    B. "I am not the best rapper in Gainesville or I do not love eating chicken wings."
    C. "If I am the best rapper in Gainesville, then I do not love eating chicken wings."
    D. The statement is not a proposition, so it cannot be negated
    E. NOTA

4. During a discussion with her parents, Robin declares, "If I get accepted to Georgia Tech, then I will study Political Science." What is the converse of her statement?

    A. "If I don't choose to study Political Science, I will not get accepted to Georgia Tech."
    B. "If I choose to study Political Science, I will get accepted to Georgia Tech."
    C. "I will not get accepted to Georgia Tech and I will not study Political Science."
    D. "I will get accepted to Georgia Tech or I will study Political Science."
    E. NOTA

5. Which of the following is logically equivalent to $\overline{P+(\overline{P}\cdot Q)}$? (Note: $\overline{A}$ denotes the complement of A, A+B denotes A OR B, $A\cdot B$ denotes A AND B.)

A. $P\cdot Q$

B. $(P\cdot Q)+(\overline{P}\cdot\overline{Q})$

C. T

D. $\overline{P}\cdot\overline{Q}$

E. NOTA

6. The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B. Which of the following functions map **N** (the set of positive integers) to the set of odd positive integers, thus showing equal cardinality?

A. $f(n)=\dfrac{n}{2}$

B. $f(n)=2n+1$

C. $f(n)=n^2-1$

D. $f(n)=2n-1$

E. NOTA

7. A set that is either finite or has the same cardinality as the set of natural numbers is called countable. A set that is not countable is called uncountable. How many of the following sets are, in fact, countable?

  i.   All possible outcomes of the Georgia State lottery
  ii.   Real numbers between -1 and 1 (not inclusive)
  iii.   Integers greater than 5 and divisible by 2
  iv.   Rational numbers between 100 and 100.1
  v.   Complex numbers, z, that satisfy $|z|\le 1$

A. 1   B. 2   C. 3   D. 4   E. NOTA

8. An algorithm for finding the maximum element in a finite sequence of integers is given below. What is the time complexity of this algorithm?

```
procedure maximum(a₁,a₂,…,aₙ: integers)
max := a₁
for i := 2 to n
   if max < aᵢ then max := aᵢ

end with max as largest value
```

A. $O(1)$
B. $O(n)$
C. $O(\log n)$
D. $O(n^2)$
E. NOTA

9. The prime factorization of 7007 can be written as $A^2\cdot B\cdot C$ with $A<B<C$. What is the value of $2A+B-C$?

A. 11   B. 17   C. -3   D. 13   E. NOTA

10. The notation $a \equiv b(\bmod m)$ indicates that a is congruent to b modulo m. What does this imply mathematically? (Note: Z represents the set of integers, N the set of natural numbers, and R the set of real numbers)

A. $\dfrac{a-b}{m} \in Z$

D. $a - b = m$

B. $\sqrt{m^2 - a - b} \in R$

E. NOTA

C. $\dfrac{a+b}{m} \in N$

11. A positive integer is called "perfect" if it equals the sum of its positive divisors other than itself. How many elements of $\{4,6,28,101,\}$ are perfect?

A. 2   B. 3   C. 4   D. 0   E. NOTA

12. What is $(37F)_{16}$ represented in binary using 12 digits?

A. 0011 1110 1111   D. 0011 0111 1111
B. 0100 1000 1100   E. NOTA
C. 0010 0111 1111

13. Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, find A*A$^\text{T}$.

A. $\begin{bmatrix} 17 & 22 & 27 \\ 22 & 29 & 36 \\ 27 & 36 & 45 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

E. NOTA

C. $\begin{bmatrix} 14 & 32 \\ 32 & 77 \end{bmatrix}$

14. Given that $A = \begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix}$, find A^{-1}.

A. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$ B. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$ C. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$ D. $\begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$ E. NOTA

15. A function $\Phi(n)$ is defined recursively for nonnegative integers n as follows:

$$\Phi(n) = \begin{cases} 1 & \text{when } n = 0 \\ n \cdot \Phi(n-1) & \text{when } n > 0 \end{cases}$$

What is the value of $\Phi(7)$?

A. 720   B. 120   C. 49   D. 64   E. NOTA

16. Among 100 people, at least how many people were born in the same month?

A. 8   B. 9   C. 10   D. 11   E. NOTA

17. How many ways are there to select a committee consisting of 3 physics professors and 4 math professors if there are 9 total physics professors and 11 total math professors?

A. 414   B. 27,720   C. 1140   D. 330   E. NOTA

18. Simplify the following expression: $\binom{n}{k-1} + \binom{n}{k}$ with $n \geq k$ and $n, k \in Z^+$

A. $\binom{n^2}{k^2-k}$ B. $\binom{n+1}{k}$ C. $n!$ D. $\binom{n}{k+1}$ E. NOTA

19. What is the probability that a bit string of length four, generated at random, contains at least two consecutive zeros given that the first bit is a zero?

A. $\frac{3}{8}$ B. $\frac{1}{4}$ C. $\frac{2}{3}$ D. $\frac{5}{8}$ E. NOTA

20. Richard invented a gambling game in an attempt to win money off of his younger cousins. He will roll a single die one time and take note of the result. If the die shows an odd number, Richard must pay $1, if the die shows a 2, nothing happens, and if the die shows anything else, Richard wins $2. What is the average amount of winnings for Richard on each roll? (Approximate to the nearest cent)

A. $0.50   B. $1.00   C. $0.17   D. $0.25   E. NOTA

21. A simple graph is shown to the right. What is the adjacency matrix for this graph with respect to the ordering of vertices A,B,C,D?
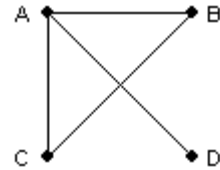
A. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$
B. $\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
C. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
E. NOTA

22. A directed graph is shown to the right. What is the adjacency matrix for this graph with respect to the ordering of vertices A,B,C,D?
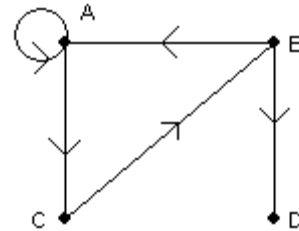
A. $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$
D. $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
E. NOTA

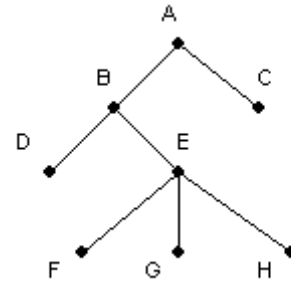C. $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

23. A simple graph is shown to the right. Does this graph have a Euler circuit?

A. Yes
B. No
C. Euler circuits do not apply to simple graphs
D. Cannot be determined
E. NOTA
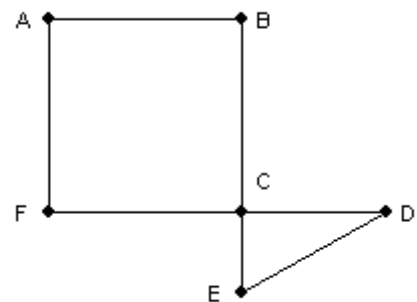
24. A rooted tree is shown to the right. In what order will the vertices
be visited by a preorder traversal?

A. A,B,D,E,F,G,H,C
B. A,B,C,D,E,F,G,H
C. D,F,G,H,E,B,A,C
D. A,B,E,C,D,F,G,H
E. NOTA

25. Suppose that a relation R on a set is represented by the matrix: $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Which of the following statements are true?

    i. R is reflexive
    ii. R is symmetric
    iii. R is antisymmetric

A. i   B. i and iii   C. ii   D. ii and iii   E. NOTA

26. A recurrence relation is given by:
$$a_0 = 0$$
$$a_1 = 1$$
$$a_n = a_{n-1} + a_{n-2}$$

What is a_8 ?

A. 8   B. 21   C. 34   D. π E. NOTA

27. How many ways are there to distribute hands of 5 cards to each of four players from a standard deck of 52 cards?

A. $\dfrac{50!}{(5!)^4 32!}$ B. $\dfrac{52!}{5!}$ C. $\dfrac{52!}{(5!)^4 32!}$ D. $\dfrac{37!}{(5!)^4}$ E. NOTA

28. A coin is biased so that the probability of heads is $\frac{2}{3}$. What is the probability that exactly four heads come up when the coin is flipped seven times, assuming that the flips are independent? (Approximate to the nearest hundredth)

A. 0.20   B. 0.31   C. 0.16   D. 0.26   E. NOTA

29. How many of the following statements are true?

      i.   if n is an odd integer, then n^2 is an odd integer

      ii.   If 3n+2 is an odd integer, then n is an odd integer

      iii. $\sqrt{2}$ is irrational

      iv.   The integer n is odd if and only if n^2 is odd

A. 0   B. 1   C. 4   D. 3   E. NOTA

30. What is the least common multiple of $2^3 3^5 7^2$ and $2^4 3^3$?

A. 2^3 B. $2^4 3^5 7^2$ C. $2^3 3^3$ D. 7^2 E. NOTA