2004 National Mu Alpha Theta Convention
Alpha Division–Number Theory Topic Test

1. **C** Since $70 = 2 \times 5 \times 7$, the largest prime divisor is 7.

2. **C** $111/3 = 37$.

3. **B** The smallest is the product of the three smallest primes: $(2)(3)(5) = 30$.

4. **A** If 0 were positive, then $(-4) \times 0 = 0$ provides a contradiction of I. The others remain true.

5. **A** Since a is even, it is divisible by 2. Since it has integer cube root and square root, it is divisible by 2^6. (Note that $a = 64$ satisfies the conditions of the problem.)

6. **B** The sum of the first n positive integers is $n(n+1)/2$. The smallest n for which this is divisible by 13 is $n = 12$.

7. **B** There are 7 multiples of 14 less than 100.

8. **B** Only 3 is a prime that is one less than a perfect square since $n^2 - 1 = (n-1)(n+1)$ always has at least 2 factors if $n > 2$.

9. **C** The principal takes $(99 - n)/6$ balloons. There are 16 multiples of 6 less than 100.

10. **A** The sum is divisible by 7 since the two given remainders sum to 7.

11. **B** The number $n! + 1$ is divisible by n for $n = 1$. For all other n, $n! + 1$ leaves a remainder of 1 when divided by n.

12. **D** Take each to the 1/100 power and we get, in order, 32, 81, 64, 25, so 5^{200} is the smallest.

13. **D** $3618 = 134 \times 27$; $938 = 134 \times 7$.

14. **B** The smallest number remaining is the product of two primes (not necessarily distinct) greater than 7, or $11^2 = 121$.

15. **A** All but 0 and 4 are easily dismissed by noting that only 00 or 44 could be repeated last 2 digits. An ending of 4444 can be dismissed by noting that any such number is of the form 16k+12, which cannot be a perfect square.

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16. **B** If $m|n$ and $n|m$, $m = n$.

17. **A** There are $120(1 - 1/2)(1 - 1/3)(1 - 1/5) = 32$ such numbers.

18. **C** Since $n^5 - 5n^3 + 4n = (n-2)(n-1)(n)(n+1)(n+2)$, we know the product is divisible by 3, 5, and 8. (Note that for $n = 3$, our product equals 120.)

19. **B** There are 60 numbers from 10 to 99 which are divisible by 2 or by 3 (or both), so the probability is $60/90 = 2/3$.

20. **C** We have $x^2 - y^2 = (x-y)(x+y) = 35$. Since $35 = 35 \times 1 = 7 \times 5$, we have two sets of solutions. $x + y = 35$, $x - y = 1$ gives us one and $x + y = 7$, $x - y = 5$ gives us the other.

21. **B** The quantity is only prime for $p = 3$, which can be shown by noting that it is divisible by 3 for all odd numbers.

22. **D** This statement is true for all pairs of positive integers.

23. **D** $420 = 42 * 10$, $294 = 42 * 7$. Therefore, $m + n = 420a + 294b = 42(10a + 7b)$.

24. **A** The solutions are $(9, 7)$ and $(4, 15)$.

25. **B** $100A + 10B + C$ is either 900 or 360, so $A + B + C = 9$.

26. **A** There's a one to one correspondence between base 3 numbers without 2's and base 2 numbers. Therefore, we interpret every base 3 number without 2's from 1_3 to $1000000_3 = 729$ as base 2 number and get $1_2 = 1$ through $1000000_2 = 64$. We omit the last, since we want the numbers less than 729, for a total of 63.

27. **C** $120 = 3 \times 2^3 \times 5$ and $40 = 2^3 \times 5$, so k must have one factor of 3. It can have anywhere from 0 to 3 factors of 2 and 0 or 1 factors of 5, so there are 8 possibilities.

28. **C** We consider each of the cases $n = 2, 3, \ldots, 9$ and find that there are 24 such pairs.

29. **B** Note that n is among the set of cycles of n. We will show that every 10-digit multiple of n that is a multiple of 11111 is such that all the members of the set of cycles of n are multiples of 11111. Let

$$n = a \cdot 10^9 + b,$$

where b is a 9-digit integer. Then, the last 'cycle' member is given by

$$10b + a = 10n - a \cdot 10^{10} + a = 10n - a(10^{10} - 1).$$

Since $10^{10} - 1 = 9999999999$, $11111|9999999999$, and $11111|n$, we know $11111|a(10^{10} - 1)$ and $11111|10n$, so $11111|10b+a$. Thus, the members of the set of cycles of n will all be divisible by 11111 if n is, and will not otherwise. Since the 10-digit multiples of n range from $90000 * 11111 + 11111 = 100001111$ to $900009 * 11111 = 9999999999$, there are $900009 - 90001 + 1 = 810009$ values of n which satisfy the problem.

30. **B** Since $f(n)$ equals the number of 1's in the binary representation of n, there are 5 numbers less than 2003 with $f(n) = 10$ (Consider a binary number with 11 digits, one of which is 0. There are 11 such numbers; 6 of these are greater than 2002.)