1. Two ordinary dice are rolled. What is the probability that the resulting sum is an even number?

   (A) 17/36 **(B)** 1/2 **(C)** 5/9 **(D)** 7/12 **(E)** NOTA

   Solution: B. The probability that each die results in an even number is 1/2, and they are independently both odd or both even.

2. Let $\mathcal{U} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Suppose an element of $\mathcal{U}$ is randomly selected. What is the probability that it belongs to $\{3, 4\} \cup \{4, 5, 6, 7\} \cup \{6, 7, 8\}$.

   (A) 0.4 **(B)** 0.5 **(C)** 0.6 **(D)** 0.7 **(E)** NOTA

   Solution: C. The set in question has 6 elements, so the probability is $6 \div 10 = 0.6$.

3. A card is randomly selected from an ordinary deck of 52 playing cards. What is the probability that it is a red face (Jack, Queen or King) card?

   (A) 3/52 **(B)** 3/26 **(C)** 3/13 **(D)** 1/2 **(E)** NOTA

   Solution: B. There are 6 red face cards, so $p = 6/52 = 3/26$

4. Suppose A, B, and C are events such that $P(A) = 1/3, P(B) = 1/2$, $P(C) = 1/4$, $P(A \cup B) = 3/4$, $P(B \cup C) = 1/2$, and $P(A \cup C) = 5/12$. Which one of the following pairs of events are disjoint (ie, mutually exclusive)?

   (A) A, B **(B)** A, C **(C)** B, C **(D)** $A \cup B, C$ **(E)** NOTA

   Solution: E. Use the formula $P(A \cap B) = P(A) + P(B) - P(A \cup B)$ to see that none of the pairs are disjoint, since the probability of their intersection is positive.

5. Suppose A, B, and C are events such that $P(A) = 1/3, P(B) = 1/2$, $P(C) = 1/4$, $P(A \cap B) = 1/6$, $P(B \cap C) = 1/6$, and $P(A \cap C) = 1/10$. Which one of the following pairs of events are independent?

   (A) A, B **(B)** A, C **(C)** B, C **(D)** $A \cap B, C$ **(E)** NOTA

   Solution: A. A and B are independent because they satisfy $P(A \cap B) = P(A) \cdot P(B)$.

6. How many of the four-digit integers can be written using exactly three digits?

   (A) 2880 **(B)** 3888 **(C)** 4000 **(D)** 4320 **(E)** NOTA

   Solution: B. If we, for the moment include those numbers that start with 0, we get $10 \cdot \binom{4}{2} \cdot 9 \cdot 8 = 4320$. But only $9/10$ of these are really four-digit numbers, so we have $9/10 \cdot 4320 = 3888$.

7. A digit is randomly selected from the set of digits comprising the decimal representation of $1/81$. What is the probability that the digit is 9?

   (A) 1/10 **(B)** 1/9 **(C)** 1/5 **(D)** 2/9 **(E)** NOTA

   Solution: B. The decimal representation of $1/81$ is given by $0.\overline{012345679}$, so the probability of 9 is $1/9$.

8. Six integers $a, b, c, d, e,$ and f are randomly selected. What is the probability that some pair of them differ by a multiple of 5?

   (A) 1/2 **(B)** 2/3 **(C)** 3/5 **(D)** 1 **(E)** NOTA

   Solution: D. There are only 5 possible remainders when an integer is divided by 5. By the Pigeonhole Principle, two of the integers must differ by a multiple of 5.

9. Three integers $a, b,$ and c are randomly selected. What is the probability that some pair of them differ by a multiple of 5?

   (A) 12/25 **(B)** 1/2 **(C)** 13/25 **(D)** 1 **(E)** NOTA

   Solution: C. The number of triples that do not have the property is $5 \cdot 4 \cdot 3 = 60$ so the probability is $1 - 60/125 = 65/125 = 13/25$.

10. A bag of 20 marbles has five red, five green, five blue, and five yellow marbles. Four marbles are randomly selected (without replacement). What is the probability that two or more marbles are the same color?

    (A) 800/969 **(B)** 4219/4845 **(C)** 864/969 **(D)** 844/969 **(E)** NOTA

    Solution: D. There are 5^4 ways to choose four different color marbles, $p = 1 - 5^4/\binom{20}{4} = 844/969$.

11. How many scoring sequences are possible if the Fortyniners won their soccer game by a score of 5 to 4 and they were never behind in the game?

    (A) 36 **(B)** 38 **(C)** 40 **(D)** 42 **(E)** NOTA

    Solution: D. Build a lattice that includes the points $(0,0), (1,0), (2,0), (.5,.5), (1.5,.5), \ldots$, so that each path starting at $(0,0)$ and ending at $(5,.5)$ represents a scoring sequence. Adding the way we do to construct Pascal's triangle produces exactly 42 scoring sequences that do not dip below the x-axis.

12. In the game *Candy*, a random cost in the range 1 cent to 1 dollar inclusive is selected. The contestant wins if he has in his pocket coins that will pay exactly the selected cost (with no change required). Steve has 2 pennies, 3 dimes and a quarter. For example, if the selected cost was 32 cents, Steve would win because he could buy the Candy bar, but if the cost was 34 cents, he would lose. What is the probability that Steve wins the game?

    (A) 0.23 **(B)** 0.28 **(C)** 0.29 **(D)** 0.30 **(E)** NOTA

    Solution: Counting 0 for the moment, there are eight groups of 3 consecutive values that can be made, for a total of 24 values. But we shouldn't count 0, so the probability is just 0.23.

13. In a carnival game, a contestant tosses a 1-inch diameter circular disk onto a grid of squares two inches on a side. The contestant wins if the disk falls entirely inside one of the squares. Given that the disk lands in the grid (that is, the center of the disk lies in one of the squares), what is the probability that the contestant wins?

    (A) 1/4 **(B)** 1/3 **(C)** 1/2 **(D)** $1/\pi$ **(E)** NOTA

    Solution: The center of the disk must be located in a 1×1 square centered inside one of the 2×2 squares, so the probability is 1/4.

14. A point P is randomly selected from the square with vertices $(1,1), (-1,1), (1,-1), (-1,-1)$. What is the probability that P is closer to $(0,0)$ than it is to $(1,1)$?

    (A) 1/8 **(B)** 3/8 **(C)** 5/8 **(D)** 7/8 **(E)** NOTA

    Solution: Draw a picture. You'll see that 7/8 the points of the square are closer to the origin.

15. Each face of three cubes is colored either B(lue) or R(ed). One of the three cubes, whose planar representations (i.e., *nets*) are given below is randomly selected and rolled. What is the probability that the color 'red' is rolled?

	B		
B	B	R	R
	B		

CUBE 1

	R		
R	R	B	B
	B		

CUBE 2

	B		
B	R	B	R
	R		

CUBE 3

(A) 4/9 **(B)** 1/2 **(C)** 5/9 **(D)** 2/3 **(E)** NOTA

Solution: A. Eight of the equally likely 18 faces are red, so the probability is $8/18 = 4/9$.

16. A point P is randomly selected from the rectangle with vertices $A = (0,0), B = (2,0), C = (2,1)$, and $D = (0,1)$. What is the probability that the angle APB is obtuse?

(A) $\frac{\pi}{4}$ **(B)** $\frac{4-\pi}{2}$ **(C)** $1-\frac{\pi}{4}$ **(D)** $\frac{\pi}{2}-1$ **(E)** NOTA

Solution: A. Draw a picture. You'll see $\angle APB$ is obtuse if and only if P lies inside the semicircle of radius 1 centered at $(1,0)$. The desired probability is therefore $\pi/2 \div 2 = \pi/4$.

17. A deck of n cards consists only of red and green cards. When two cards are selected simultaneously and without replacement, the probability that they are both green is twice the probability that they are both red. What is the smallest possible value of n?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** NOTA

Solution: B. The first pair of triangular numbers with the desired relationship is 3 and 6. Thus there could be three red and four green cards.

18. A particle moves among four states probabilistically according to the matrix given. For example, if the particle is in state D, it moves to state B with probability 1/3. What is the probability that, starting in A, the particle is in D after exactly 3 moves?

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1/2 & 1/2 & 0 \\ B & 1/2 & 0 & 0 & 1/2 \\ C & 1/2 & 0 & 0 & 1/2 \\ D & 0 & 1/3 & 1/3 & 1/3 \end{array}.$$

(A) 1/12 **(B)** 1/6 **(C)** 1/3 **(D)** 5/12 **(E)** NOTA

Solution: B. There are just two paths from A to D in three moves and they occur with probabilities $1/2 \cdot 1/2 \cdot 1/3$ and $1/2 \cdot 1/2 \cdot 1/3$ for a total of 1/6.

19. The faces of a cube are colored red and blue, one at a time, with equal probability. What is the probability that the resulting cube has a vertex P such that all three faces containing P are colored red?

(A) 1/4 **(B)** 5/16 **(C)** 27/64 **(D)** 1/2 **(E)** NOTA

Solution: C. Among all 64 possible cubes, those with 6 red, or with 5 red faces must have such a vertex. Among the 15 with 4 red faces, 12 have such a vertex and among the 20 with 3 red faces, exactly 8 have such a vertex. So the number of 'good' cubes is 27, and the probability is 27/64.

20. A real number θ is randomly selected from the interval $[0, \pi)$. What is the probability that $(\sin\theta + \cos\theta)^2 \leq 1$?

(A) 0 **(B)** 1/2 **(C)** $1/\pi$ **(D)** 1 **(E)** NOTA

Solution: B. Since $(\sin\theta + \cos\theta)^2 = \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta \leq 1$ if and only if $2\sin\theta\cos\theta = \sin(2\theta) \leq 0$ which is true for half the values of θ in the interval.

21. How many distinguishable cubes can be built using a supply of blue and red faces?

(A) 6 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** NOTA

Solution: There is just one type of cube with none, one, five and six red faces, and there are two cubes with two, three, and four red faces, for a total of 10 altogether.

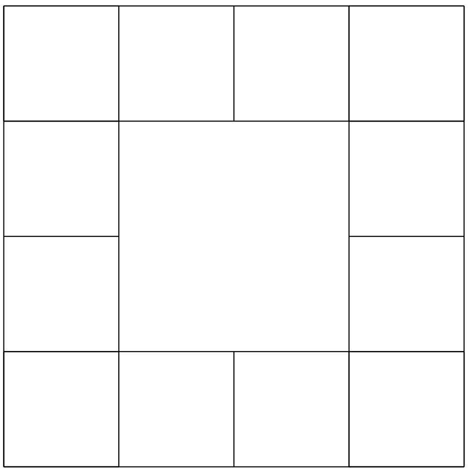
22. Three sets of Mu Alpha Theta students, M, A, and T satisfy the following properties: $|MAT| = |MA\overline{T}| = |M\overline{A}T| = |\overline{M}AT| = |M\overline{A}\,\overline{T}|$ and $|M| = 20$, $|A| = 17$, and $|T| = 19$. What is $|M \cup A \cup T|$? Recall that UV refers to $U \cap V$.

    (A) 23 **(B)** 25 **(C)** 27 **(D)** 31 **(E)** NOTA

    Solution: The four sets $MAT, MA\overline{T}, M\overline{A}T$, and $M\overline{A}\,\overline{T}$ partition M. Since they have the same number of members, each must have $20/4 = 5$ members. Since $|\overline{M}AT| = 5$, it follows that $|\overline{M}\,\overline{A}T| = 4$ and $|\overline{M}A\overline{T}| = 2$. So $|M \cup A \cup T| = 5 + 5 + 5 + 5 + 5 + 2 + 4 = 31$.

23. A square whose edges are a subset of the gridlines in the figure shown is randomly selected. The smallest squares in the grid are unit squares. Let m/n denote the probability that the selected square has area at least 4, and m/n is reduced. What is $m + n$?

    (A) 1/4 **(B)** 1/3 **(C)** 1/2 **(D)** 5/17 **(E)** NOTA

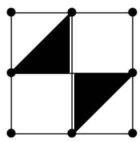


Solution: B. We can count the number with area at least 4 by noting that there is 1 of area 16; 4 with area 9; and 1 with area 4 for a total of 6. There are 12 unit squares so there are $12 + 6 = 18$ squares altogether. Therefore, the probability is $6/18 = 1/3$.

24. Three points are randomly selected on the circumference of a circle. What is the probability that the triangle having these points as vertices contains the center of the circle?

    (A) 1/4 **(B)** 1/3 **(C)** 1/2 **(D)** 2/3 **(E)** NOTA

    Solution: A. We can model the problem as follows. The first point can be assumed to be the point $(1, 0)$. Let x represent the counterclockwise distance from the first point to the second point, and y the counterclockwise distance from the first point to the third point. Then the point (x, y) in the square shown below represents a choice of the three points. The shaded pair of triangles represents the region of the square corresponding to when the triangle contains the center of the circle.



25. A coin is biased so that the probability of landing heads exactly twice when flipped three times is 2/9. Assuming that the probability that the coin lands heads is rational, what is the probability that all three flips land heads?

    (A) 1/27 **(B)** 1/9 **(C)** 1/3 **(D)** 1/2 **(E)** NOTA

    Solution: We first solve the equation $3p^2(1-p) = 2/9$ to get $p = 1/3$, where p is the probability of heads. Thus $p^3 = 1/27$. The other two roots of $3p^2(1-p) = 2/9$ are irrational.

26. An urn consists entirely of blue and red marbles. When two marbles are randomly selected (without replacement), the probability that they are different colors is 10/21. Given that the number of blue marbles is an integer multiple of the number of red marbles, which of the following could be the number of marbles in the urn?

    (A) 7 **(B)** 14 **(C)** 15 **(D)** 16 **(E)** NOTA

    Solution: C. Let r and b denote the number of red and blue marbles respectively. Since $\frac{rb}{\binom{r+b}{2}} = \frac{10}{21}$, it follows that either 5 divides r or 5 divides b. Letting $r = 5$ for the moment, we can solve $21b = (b+5)(b+4)$ for b to get $b = 2$ or $b = 10$. The hypothesis implies that $b = 10$.

27. In a round-robin holiday basketball tournament, four teams play one another on three successive days. Each team plays the other three. Each game is evenly matched and there are no ties. What is the probability that after three days, all four teams have different records?

    (A) 1/8 **(B)** 1/4 **(C)** 3/8 **(D)** 1/2 **(E)** NOTA

    Solution: C. There are $2^6 = 64$ different tournaments, and these can be classified by the records obtained by the teams: The record $3-0, 2-1, 1-2, 0-3$ occurs in 24 of the tournaments. The other records have some tied teams: $3-0, 1-2, 1-2, 1-2$ occurs 8 times; $2-1, 2-1, 2-1, 0-3$ occurs 8 times, and $2-1, 2-1, 1-2, 1-2$ occurs 24 times.

28. A four-digit number *abcd* is called 'non-decreasing' if each digit after the first one is at least as large as the one to its left. For example, 1336 is non-decreasing. How many four-digit non-decreasing numbers are there? Note that 0123 is not a four-digit number.

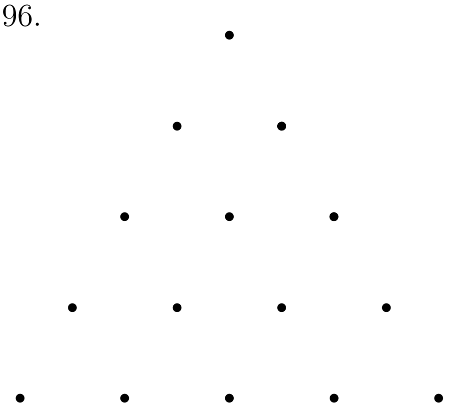
    (A) 126 **(B)** 210 **(C)** 330 **(D)** 495 **(E)** NOTA

    Solution: D. There are several types of non-decreasing numbers and we count them by type. We can denote these types by *abcd*, *aabc*, *abbc*, *abcc*, *abbb*, *aabb*, *aaab*, and *aaaa* where $a < b < c < d$. There are $\binom{9}{4} = 126$ of the first type, $\binom{9}{3} = 84$ of each of the types with three distinct digits, $\binom{9}{2} = 36$ of each of the two-digit types, and 9 of the type *aaaa*, for a total of 495. There is also a very slick method that show this number to be $\binom{12}{4} = 495$ directly.

29. How many right triangles have all three of their vertices among the 15 dots of the hexagonal lattice?

    (A) 30 **(B)** 60 **(C)** 66 **(D)** 96 **(E)** NOTA

    Solution: D. We can use the 3-wise symmetry of the figure to cut down on the calculations. The three center points are the vertices of ten right-angled triangles. That is, there are 10 right triangles that have each of these points as the vertex at which the right angle occurs. There are 12 triangles with the center outside points as the location of the right angle. The six points that are not extreme points and not center points are vertices of five right triangles where the right angle occurs at that point. Finally there are no right triangles that have the right angle occurring at an extreme point. So the total number is $3 \cdot 10 + 3 \cdot 12 + 6 \cdot 5 = 96$.



30. Charlie plays the game *Magcam* by repeatedly rolling a standard die. Charlie rolls until a 1 appears. His payoff in dollars is the number of rolls of the die. For example, the sequence $2, 4, 3, 1$ would earn Charlie \$4. What is Charlie's expected gain from playing this game? In other words, what would be a fair price to pay for playing this game once?

    (A) \$2 **(B)** \$3 **(C)** \$4 **(D)** \$6 **(E)** NOTA

    Solution: We need to find the sum $S = 1 \cdot 1/6 + 2 \cdot 5/6 \cdot 1/6 + 3 \cdot (5/6)^2 \cdot 1/6 + \cdots$. Multiplying both sides by $5/6$ and subtracting gets the geometric series $S/6 = 1/6[1 + 5/6 + (5/6)^2 + \ldots]$, from which it follows that $S = 6$.

tiebreaker 1 Row zero of Pascals triangle consists of a single entry, row one has two entries, etc. How many of the entries in rows 0 through 1023 of Pascal's triangle are odd numbers?

Solution: Notice that the number of odd entries in the first two rows (rows 0 and 1) is 3, and the number in the first four rows (rows 0 through 3) is 9. Observe that the number of odd integers among the first 2^n rows is 3^n. Thus the rows 0 through 1023 contain 3^{10} odd entries.

tiebreaker 2 For how many of the first 100 is positive integers n does the decimal representation of $n!$ ends with an even number of zeros?

Solution: The number $n!$ ends with $\lfloor n/5 \rfloor + \lfloor n/25 \rfloor$ zeros. The numbers for which the value $\lfloor n/5 \rfloor + \lfloor n/25 \rfloor$ is even are $1, 2, 3, 4, 10, 11, 12, 13, 14, 20, 21, 22, 23, 24, \ldots, 100$, and there are 60 of these.