Alpha State Bowl Solutions

1. A triangle with angle u and opposite and adjacent sides of 1 and $\sqrt{3}$ would have hypotenuse 2. The cosine of this angle would be $\boxed{\dfrac{\sqrt{3}}{2}}$.

2. The probability that it won't rain is $1 - \frac{3}{4} = \frac{1}{4}$. The probability of no rain three days in a row is $\left(\frac{1}{4}\right)^3 = \boxed{\dfrac{1}{64}}$.

3. Taking 10 BC as "year -10," we find that his life spanned $60 - -10 = 70$ years — however, there is no year zero, so he must have been $\boxed{69}$ years old when he died.

4. Let U equal the time (in hours) Ulysses would spend working alone and V equal the time Victor would spend working alone. $U = 36$ and $V = 18$ since he works twice as fast. $\frac{1}{18} + \frac{1}{36} = \frac{1}{T}$ where T is their total time spent when working together. $T = \boxed{12}$.

5. Using the power reducing formula, we find that $f(t) = 4\cos^2(t) = 4\left(\frac{1+\cos(2t)}{2}\right) = 2 + 2\cos(2t)$. The maximum value occurs when $\cos(2t) = 1$, when $f(t) = 4$. The minimum value occurs when $\cos(2t) = -1$, when $f(t) = 0$. $|4 - 0| = \boxed{4}$.

6. If $BA = I$, then $B = A^{-1}$. $|B| = \frac{1}{|A|}$. $|A| = 6 \cdot 2 - (-1)(-3) = 9$, so $|B| = \frac{1}{9}$.

7. We are given that $f(\sqrt{2}) = f(\sqrt{7})$. The vertex of a parabola must occur at the midpoint of two points with the same height, so the x-coordinate is $\boxed{\dfrac{\sqrt{2}+\sqrt{7}}{2}}$.

8. The maximum value will occur when the two integers are as close together as possible. $25 \div 2 = 12.5$ so the integers should be 12 and 13; $12 \cdot 13 = \boxed{156}$.

9. The first 25 positive even numbers sum to $25^2 + 25$. $\sqrt{25^2 + 25} = 5\sqrt{25+1} = \boxed{5\sqrt{26}}$.

10. $3 \mod 4 \equiv -1$ so we have $(-1)^{2005} \equiv -1 \mod 4 \equiv \boxed{3} \mod 4$.

11.

$$\begin{aligned} \cos(t) + \sin(t) &= 0 \\ \sin(t) &= -\cos(t) \\ \tan(t) &= -1 \\ t &= \frac{3\pi}{4}, \frac{7\pi}{4} \end{aligned}$$

$\frac{3\pi}{4} + \frac{7\pi}{4} = \boxed{\dfrac{5\pi}{2}}$.

12. 2005π is coterminal with π, so we have $\sin(\pi) + \cos(\pi) + \tan(\pi) = 0 - 1 + 0 = \boxed{-1}$.

13. The infinite geometric series formula works for $|r| < 1$ even when r is complex. Letting $z = 3 - i$, the sum is $\frac{z^{-1}}{1-z^{-1}} = \frac{1}{z-1} = \frac{1}{2-i} = \boxed{\dfrac{2+i}{5}}$.

14. $\sum_{n=1}^{2005}(1 - 2n) = 2005 - (2005^2 + 2005) = -2005^2 = \boxed{-4020025}$.

15. The geometric mean of a set of n numbers is the nth root of the product of the numbers. $(\sqrt{2}+1)(\sqrt{2}-1)(4)(2\sqrt{2}) = 8\sqrt{2} = 2^3 \cdot 2^{\frac{1}{2}} = 2^{\frac{7}{2}}$. The fourth root is $2^{\frac{7}{8}}$ so $n = \boxed{\dfrac{7}{8}}$.

16. Use the half-angle formula for tangent. $\tan(\frac{\pi}{12}) = \frac{1-\cos(\frac{\pi}{6})}{\sin(\frac{\pi}{6})} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \boxed{2-\sqrt{3}}$.

17. Using the power reducing formula, we find that $y = 1 - 5\sin^2(3x-2) = 1 - \frac{5}{2}(1-\cos(6x-4))$. The coefficient of the x inside the cosine is 6, making the period $\frac{2\pi}{6} = \boxed{\dfrac{\pi}{3}}$.

18. $X = 2^1 \cdot 2^2 \cdots 2^{27} = 2^{1+2+\ldots+27} = 2^{378} = 4^{189}$. A number in base 4 has as many zeros as factors of 4, so $\boxed{189}$.

19. We know that the polynomial has one real root, so the discriminant must be zero. The linear coefficient, b, is unknown, but we know $a = 8$ and $c = \frac{1}{2}$.

$$\begin{aligned} b^2 - 4 \cdot 8 \cdot \frac{1}{2} &= 0 \\ b^2 &= 16 \\ b &= \pm 4 \end{aligned}$$

The root will be $-\frac{b}{2a} = \mp\frac{1}{4}$. The problem asks for the largest so the root is $\boxed{\dfrac{1}{4}}$.

20. The altitudes of the equilateral triangle intersect at the center of its inscribed circle. The radius is $\frac{1}{3}$ of the length of the altitude, or $\frac{1}{3} \cdot \frac{9\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$. The area of the bull's eye is then $\frac{9\pi}{4}$. The area of the entire board (the triangle) is $\frac{9^2\sqrt{3}}{4} = \frac{81\sqrt{3}}{4}$. The probability of landing in the circle is the ratio of these two areas, $\boxed{\dfrac{\pi\sqrt{3}}{9}}$.

21. Let x be the number of true/false questions answered correctly and y be the number of free response questions. Then $2x + 5y = 30$. The only possible solutions given the constraints are $(5,4)$ and $(10,2)$. He could've answered at minimum $\boxed{9}$ questions correctly.

22. $\arcsin(x) + \arccos(x) = \frac{\pi}{2}$ and $\arctan(x) + \text{arccot}(x) = \frac{\pi}{2}$ for any value of x. The expression simplifies to $\sec\left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \boxed{-1}$.

23. The total surface area is the area of all four triangles plus the area of the square base. $4 \cdot \frac{\sqrt{3}^2\sqrt{3}}{4} + \sqrt{3}^2 = \boxed{3\sqrt{3}+3}$.

24. We start with $4:6$ ratio of gasoline to oil. We wish to obtain a $5:3 = 10:6$ ratio. Thus we will need to add $\boxed{6}$ gallons of gasoline.

25. In standard form, the equation is $x^2 - \frac{y^2}{4} = 1$. The slope of the asymptotes is $\pm\frac{b}{a} = \pm 2$. $m^2 = 2^2 = \boxed{4}$.

26. The area of an annulus is $\pi(R^2 - r^2)$ where R is the outer radius and r is the inner radius. $\pi\left(\left(\frac{9}{4}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2\right) = \pi(\frac{81}{16} - \frac{3}{4}) = \boxed{\dfrac{69\pi}{16}}$.

27. There are 4 possible sequences: (1,2,3), (2,3,4), (3,4,5), and (4,5,6). Each of these has 6 possible arrangements for a total of 24 outcomes. There are 216 possible outcomes when rolling 3 dice, so the probability is $\frac{24}{216} = \boxed{\dfrac{1}{9}}$.

28. The focus and the directrix of a parabola are equidistant from the vertex. The focus is $\frac{5}{2} - 2 = \frac{1}{2}$ units horizontally and $5 - 4 = 1$ unit vertically from the vertex, so the directrix must pass through the point $(\frac{5}{2} + \frac{1}{2}, 5 + 1) = (3, 6)$. The directrix runs perpendicular to the line connecting the vertex and focus. This line has a slope of $\frac{1}{\frac{1}{2}} = 2$ so the directrix has a slope of $-\frac{1}{2}$. The equation of the directrix is given by $-\frac{1}{2}(x - 3) = y - 6$. The x-intercept is the point where $y = 0$. Plugging in and solving, we obtain $x = 15$ for the x-coordinate. The x-intercept is $\boxed{(15, 0)}$.

29. First multiply through by $(x^3 - 1)$ to clear the fractions:

$$\begin{aligned} -3 &= (x+2)(x-1) + C(x^2 + x + 1) \\ -3 &= x^2 + x - 2 + Cx^2 + Cx + C \\ -3 &= (C+1)x^2 + (C+1)x + (C-2) \end{aligned}$$

It is clear that C must be $\boxed{-1}$ in order to satisfy the equation.

30. Rewriting in standard form, we obtain $\frac{y^2}{2} - \frac{x^2}{4} = 1$. Eccentricity is the ratio of the distance between the foci to the distance between the vertices, or $\frac{c}{b}$ in this case. $c = \sqrt{2+4} = \sqrt{6}$ and $b = \sqrt{2}$, so $e = \sqrt{\frac{6}{2}} = \boxed{\sqrt{3}}$.

31. $\sum_{n=0}^{x} n^3 = \left[\frac{x(x+1)}{2}\right]^2$

$\sum_{n=0}^{100} 4n^3 = 100^2 \cdot 101^2 = \boxed{102010000}$

32. Use polynomial long division to reformulate $f(x)$ as a linear polynomial plus a remainder term:

$$\begin{array}{rr|rrr} & & & 2x & +8 \\ \hline x^2 & -4x & 2x^3 & & +1 \\ & & -(2x^3 & -8x^2) & \\ \hline & & & 8x^2 & +1 \end{array}$$

We can stop here as we have found the equation of the slant asymptote, $y = 2x + 8$. $(a, b) = (2, 8)$ so $a + b = \boxed{10}$.

33. Rearrange the terms to obtain $(x^3 - 4x) - (x^2 - 4) = (x - 1)(x^2 - 4) = (x - 1)(x - 2)(x + 2)$. The roots are $(a, b, c) = (-2, 1, 2)$. $3a - b + 6c = -6 - 1 + 12 = \boxed{5}$.

34. First, we will reformulate $Y(n)$ in a finite form:

$$\begin{aligned} Y(n) &= \sqrt{n + Y(n)} \\ Y(n)^2 &= n + Y(n) \\ Y(n)^2 - Y(n) - n &= 0 \\ Y(n) &= \frac{1 \pm \sqrt{1+4n}}{2} \end{aligned}$$

We can discard the non-positive solution since the equation involves addition of all positive numbers. If $Y(n) = n$, we can set up the equation as follows:

$$\begin{aligned}
\frac{1+\sqrt{1+4n}}{2} &= n \\
1+\sqrt{1+4n} &= 2n \\
\sqrt{1+4n} &= 2n-1 \\
1+4n &= 4n^2-4n+1 \\
4n^2-8n &= 0 \\
4n(n-2) &= 0 \\
n &= 0, 2
\end{aligned}$$

Since $n > 0$, the only solution is $n = \boxed{2}$.

35. The harmonic mean of a set of numbers is the reciprocal of the average of the reciprocals of the numbers.

    The average of the reciprocals of the numbers is $\frac{1}{4}\left(1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}\right) = \frac{1}{4}\left(\frac{144+36+16+9}{144}\right) = \frac{205}{576}$. The reciprocal of this is $\boxed{\dfrac{576}{205}}$.

36. We can imagine the rhombus as the union of four right triangles within the ellipse, with legs a and b, the axes of the ellipse. The area of the rhombus is $R = 4 \cdot \frac{ab}{2} = 2ab$ and the area of the ellipse is $E = \pi ab$. $\frac{E}{R} = \boxed{\dfrac{\pi}{2}}$. (Note that the exact values of a and b do not matter.)

37. $\log_5(.05) = \log_5\left(\frac{1}{20}\right) = \log_5(1) - \log_5(20) = -\frac{\log(20)}{\log(5)} = -\frac{\log(10)+\log(2)}{\log(10)-\log(2)} = \frac{-1-\log(2)}{1-\log(2)}$.

    $\log(c) = x$ and $\log_c(2) = v$ so $\log(2) = v\log(c) = vx$. Thus, the final answer will be $\boxed{\dfrac{-1-vx}{1-vx}}$ (or $\boxed{\dfrac{vx+1}{vx-1}}$).

38. Draw two adjacent diagonals of the octagon to form an isosceles triangle with legs R (the radius of the circumscribed circle), altitude $\sqrt{2}$ (the apothem), and vertex angle $\frac{360^\circ}{8} = 45^\circ$. The base angles are $\frac{1}{2} \cdot (180-45)^\circ = \frac{135}{2}^\circ$. Thus, $R = \frac{\sqrt{2}}{\sin\left(\frac{135}{2}^\circ\right)}$. By the half-angle formula, we can simplify this to $\frac{2}{\sqrt{1-\cos(135^\circ)}} = \frac{2}{\sqrt{1+\frac{\sqrt{2}}{2}}}$. The area of the circle is $\pi \cdot \frac{4}{1+\frac{\sqrt{2}}{2}} = \pi \cdot \frac{4\sqrt{2}}{\sqrt{2}+1} = \pi \cdot \left(4\sqrt{2}(\sqrt{2}-1)\right) = \boxed{(8-4\sqrt{2})\pi}$.

39. $65,537 = 2^{16}+1$. The number of diagonals of an n-sided polygon is given by $D = \frac{n(n-3)}{2}$. $D = \frac{1}{2}\left((2^{16}+1)(2^{16}-2)\right) = \frac{1}{2}\left(2^{32}-2^{17}+2^{16}-2\right) = \frac{1}{2}\left(2^{32}-2^{16}-2\right) = 2^{31}-2^{15}-1$. D is slightly less than 2^{31}, so $\log_2(D)$ is just under 31. $\lceil \log_2(D) \rceil$ is therefore $\boxed{31}$.

40. We are given that $\ln(262,537,412,640,768,744) \approx \pi\sqrt{163}$. $\sqrt{163}$ is just over $\sqrt{162} = 9\sqrt{2} \approx 9 \cdot 1.414 \approx 12.7$. $\pi \cdot 12.7 \approx 3.14 \cdot 12.7 \approx 39.9$, which is $\boxed{40}$ to the nearest integer.