Mu Sequences and Series Solutions

1. The sum of the first n positive even numbers is $n^2 + n$.
   $320^2 + 320 = \boxed{102720}$.

2. The first person shakes hands with the 39 others and leaves. The second person shakes hands with the remaining 38 and leaves, and so on. $39 + 38 + 37 + ... + 1 = \frac{39}{2}(1 + 39) = \boxed{780}$.

3. $a_1 = 6$, $r = \frac{2}{6} = \frac{1}{3}$, so $S = \frac{6}{1-\frac{1}{3}} = \boxed{9}$.

4. $4 + 8 + \ldots + 100 = \frac{25}{2}(4 + 100) = \boxed{1300}$.

5. $r = \frac{a_2}{a_1} = \frac{\frac{5}{8}}{\frac{5}{7}} = \boxed{\frac{7}{8}}$.

6. $d = a_2 - a_1 = 3x^2 + 1 - x^2 = \boxed{2x^2 + 1}$.

7. $a_1 = -50$, $d = -65 - (-50) = -15$, so $a_{40} = -50 + 39 \cdot (-15) = \boxed{-635}$.

8. $a_1 = \frac{2}{5}$, $r = 2$, so $S_{10} = \frac{2}{5}(2^{10} - 1) = \boxed{\frac{2046}{5}}$.

9. $a_1 = 8$, $d = 7 - 8 = -1$, and $n = \frac{-20-8}{-1} + 1 = 29$, so $S_{29} = \frac{29}{2}(8 - 20) = \boxed{-174}$.

10. The number of rooms R on floor n is given by $R(n) = 4n + 1$.
    $\sum_{n=1}^{20}(4n + 1) = 4 \cdot \frac{20(1+20)}{2} + 20 = \boxed{860}$.

11. The nth triangular number is the same as the sum of the first n positive integers. $T_{41} = \frac{41(1+41)}{2} = \boxed{861}$.

12. The Fibonacci sequence is $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \boxed{377}, \ldots$.

13. $a_6 = \frac{10-6}{6^2+3} = \boxed{\frac{4}{39}}$.

14. As $n \to \infty$, $a_{n-1} \to a_n \to L$, so replace both terms with L:

$$\begin{aligned} L &= \sqrt{7 + \frac{6}{L}} \\ L^2 &= 7 + \frac{6}{L} \\ L^3 &= 7L + 6 \\ L^3 - 7L - 6 &= 0 \\ (L+1)(L-3)(L+2) &= 0 \\ L &= -1, -2, 3 \end{aligned}$$

    L cannot be negative since $a_n > 0$, so the limit must be $\boxed{3}$.

15. $\sum_{n=1}^{k}(n \cdot n!) = (k + 1)! - 1$, so $\sum_{n=2}^{700}(n \cdot n!) = 701! - 1 - 1 = \boxed{701! - 2}$.

16. Consider any series $\sum a_n$. $\sum a_n$ is absolutely convergent if $\sum a_n$ and $\sum |a_n|$ both converge. $\sum a_n$ is said to be conditionally convergent if $\sum a_n$ converges, but not $\sum |a_n|$. In this case, the series and its absolute value both converge; the denominator n^n becomes far greater than the numerator $(-2)^n \cdot n!$ for large values of n (use the ratio test to prove this). Therefore the series is $\boxed{\text{absolutely convergent}}$.

17. The geometric mean of a set of n numbers is equal to the nth root of the product of the numbers.

    Consider the geometric sequence with n terms, common ratio r, and first term a. The product of all terms is $a \cdot a \cdot r \cdot a \cdot r^2 \cdots a \cdot r^{n-1} = a^n \cdot r^{0+1+\ldots+(n-1)} = a^n \cdot r^{\frac{n(n-1)}{2}}$. The nth root of the product is $\boxed{a \cdot r^{\frac{n-1}{2}}}$.

18. Use the ratio test to find values of x for which the series converges:

$$\lim_{n\to\infty} \left| \frac{x^{2n+2} \cdot (n+1)^2}{3^{n+1} \cdot (n+1)!} \cdot \frac{3^n \cdot n!}{x^{2n} \cdot n^2} \right| < 1$$

$$\lim_{n\to\infty} \left| \frac{x^2 \cdot \left(\frac{n+1}{n}\right)^2}{3 \cdot (n+1)} \right| < 1$$

    The limit approaches 0 for large values of n, so the series converges for any value of x. Thus $r = \boxed{\infty}$.

19.

$$\begin{aligned} \frac{1}{n^2+1} &< \frac{1}{10000} \\ n^2+1 &> 10000 \\ n^2 &> 9999 \\ n^2 &> \sqrt{9999} \end{aligned}$$

    $\sqrt{9999}$ is just less than 100 so $|A_n|$ first becomes less than .0001 when $n = \boxed{100}$.

20. We are given that $\sum a_n = A$ and $b_n < a_n$ when $n > N$. By direct comparison, $\boxed{\sum_{n=N+1}^{\infty} b_n \text{ must converge}}$, but not necessarily to A. Consider $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^3}$. b_n is clearly less than a_n but the sums differ. Nor must the sum of *all terms* converge to a value between 0 and A. Consider $a_n = \frac{1}{n^2}$ and $b_n = \frac{1}{n^2+1}$ for $n > N$ and $b_n = A$ for $b_n \leq N$, $N > 1$. Obviously the sum of b_n for all terms greater than N is less than A, but if each preceding term is A or larger, the sum of the entire series grows larger than A.

21. Consider the series $\sum_{n=0}^{\infty} x^n = (1-x)^{-1}$ (for $|x| < 1$). Take the derivative of both sides (with respect to x) to obtain $\sum_{n=0}^{\infty} n \cdot x^{n-1} = (1-x)^{-2}$. Multiply by x on both sides to obtain $\sum_{n=0}^{\infty} n \cdot x^n = x(1-x)^{-2}$. Take the derivative again to obtain $\sum_{n=0}^{\infty} n^2 \cdot x^{n-1} = (1-x)^{-2} + 2x(1-x)^{-3}$. Finally, multiply by x to obtain $\sum_{n=0}^{\infty} n^2 \cdot x^n = x(1-x)^{-2} + 2x^2(1-x)^{-3}$.

    To evaluate $\sum_{n=0}^{\infty} \frac{1+n+n^2}{5^n}$ we simply add together the formulas obtained for the sum of x^n, $n \cdot x^n$ and $n^2 \cdot x^n$ for $x = \frac{1}{5}$.

    $(1-x)^{-1} + x(1-x)^{-2} + x(1-x)^{-2} + 2x^2(1-x)^{-3} = \frac{5}{4} + 2 \cdot \frac{1}{5} \cdot \left(\frac{5}{4}\right)^2 + 2 \cdot \left(\frac{1}{5}\right)^2 \left(\frac{5}{4}\right)^3 = \boxed{\frac{65}{32}}$.

22. $\sum_{n=1}^{\infty} n \cdot x^n = \sum_{n=2}^{\infty} (n-1)x^{n-1} = \sum_{n=2}^{\infty} n \cdot x^{n-1} - \sum_{n=2}^{\infty} x^{n-1} = \frac{d}{dx}\left(\sum_{n=2}^{\infty} x^n\right) - \frac{x}{1-x} = \frac{d}{dx}\left(\frac{x^2}{1-x}\right) - \frac{x}{1-x} = \frac{2x(1-x)+x^2}{(1-x)^2} - \frac{x}{1-x} = \frac{2x-x^2-x+x^2}{(1-x)^2} = \boxed{\frac{x}{(1-x)^2}}$.

23. $\sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)^n}{n!} = e^{\frac{1}{2}} = \boxed{\sqrt{e}}$.

24. The integral test guarantees convergence of a series if the integral converges to a finite value. However, the value it converges to has no bearing on the sum of the series itself. We know then that $\boxed{\text{the series converges to some finite value } S \text{ (not necessarily } \frac{2\pi\sqrt{3}}{9})}$.

25.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{(x-1)^{2n+2}}{4^{n+1}}\cdot\frac{4^n}{(x-1)^{2n}}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{(x-1)^2}{4}\right| &< 1\\
\left|\frac{(x-1)^2}{4}\right| &< 1\\
(x-1)^2 &< 4\\
|x-1| &< 2
\end{aligned}$$

Thus $-1 < x < 3$. We must now test the endpoints. When $x = -1$, we end up with $\sum(-1)^n$ which obviously diverges. When $x = 3$, we have $\sum 1$ which also diverges. Therefore, the interval of convergence is $\boxed{x \in (-1, 3)}$.

26. Consider the series $\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x$. Take the derivative of both sides (with respect to x) to obtain $\sum_{n=0}^{\infty}\frac{n\cdot x^{n-1}}{n!} = e^x$. Multiply by x on both sides to obtain $\sum_{n=0}^{\infty}\frac{n\cdot x^n}{n!} = x\cdot e^x$. Take the derivative again to obtain $\sum_{n=0}^{\infty}\frac{n^2\cdot x^{n-1}}{n!} = (x+1)\cdot e^x$. Multiply by x to obtain $\sum_{n=0}^{\infty}\frac{n^2\cdot x^n}{n!} = (x^2+x)\cdot e^x$. Take the derivative once more to obtain $\sum_{n=0}^{\infty}\frac{n^3\cdot x^{n-1}}{n!} = (x^2+3x+1)\cdot e^x$. Finally, multiply by x to obtain $\sum_{n=0}^{\infty}\frac{n^3\cdot x^n}{n!} = (x^3+3x^2+x)\cdot e^x$.

Using the results from above for $x = 1$, we find that $\sum_{n=0}^{\infty}\frac{n^2+3n^3}{n!} = (1^2+1)\cdot e^1 + 3\cdot(1^3+3\cdot 1^2+1)\cdot e^1 = 17e$. The sum we are evaualting actually starts two terms later than this one, so we must subtract those off to obtain $\boxed{17e-4}$.

27. $z = \frac{4\sqrt{10}}{125} = \frac{2^2\cdot 2^{\frac{1}{2}}\cdot 5^{\frac{1}{2}}}{5^3} = \left(\frac{2}{5}\right)^{\frac{5}{2}}$.

$$\begin{aligned}
z^{z^{z^{\cdots}}} &= L\\
z^L &= L\\
z &= L^{\frac{1}{L}}
\end{aligned}$$

$z = L^{\frac{1}{L}} = \left(\frac{2}{5}\right)^{\frac{5}{2}}$ so $L = \boxed{\frac{2}{5}}$.

28. $L = n^2\ln(n)$ is monotonically increasing, so the smallest positive term is the first term after $L = 0$. $L = 0$ when $n = 1$, so $L(2) = 4\ln(2) = \boxed{\ln(16)}$ is the smallest positive term.

29. The Maclaurin series for sin is $\sin(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots$. The first three terms of $\frac{\sin(x)}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120}$.

$\int_0^6 1 - \frac{x^2}{6} + \frac{x^4}{120} = x - \frac{x^3}{18} + \frac{x^5}{600}\Big|_0^6 = 6 - \frac{6^3}{18} + \frac{6^5}{600} = 6 - 12 + \frac{324}{25} = \boxed{\frac{174}{25}}$.

30. Drop an altitude from angle C intersecting with side AB at point D. Now draw a line segment perpendicular to BC starting from D ending at point E on side BC. Draw another line segment perpendicular to AB ending at point F on AB. These new line segments form the path that the ant walks along. All right triangles formed by the new line segments are similar to each other.

Extending the pattern of line segments indefinitely, we note that the lengths of the segments form two geometric series, one with terms $\overline{AC}, \overline{DE}, \ldots$ and another with terms $\overline{CD}, \overline{EF}, \ldots$. $\overline{AC} = 6\tan(15^\circ)$, $\overline{CD} = 6\sin(15^\circ)$, $\overline{DE} = 6\sin(15^\circ)\cos(15^\circ)$, and $\overline{EF} = 6\sin(15^\circ)\cos^2(15^\circ)$. This makes the common ratio of the both series $r = \cos^2(15^\circ)$. The sum of these two infinite geometric series is $S = \frac{\overline{AC}+\overline{CD}}{1-r} = \frac{6\tan(15^\circ)+6\sin(15^\circ)}{1-\cos^2(15^\circ)} = 6 \cdot \frac{\tan(15^\circ)+\sin(15^\circ)}{\sin^2(15^\circ)} = \frac{6}{\sin(15^\circ)\cos(15^\circ)} + \frac{6}{\sin(15^\circ)}$.

By the difference formula, $\sin(15^\circ) = \sin(45^\circ - 30^\circ) = \frac{\sqrt{6}-\sqrt{2}}{4}$, so $S = \frac{6}{\frac{1}{2}\sin(30^\circ)} + \frac{24}{\sqrt{6}-\sqrt{2}} = 24 + \frac{24\cdot(\sqrt{6}+\sqrt{2})}{4} = \boxed{24 + 6\sqrt{6} + 6\sqrt{2}}$.