2008 National MAΘ Convention — Alpha Ciphering — Solutions

1. **36** $f(x) = 3x^2 - 14x - 24 = (3x+4)(x-6)$, roots are $-\frac{4}{3}$ and 6. A = $6 - \frac{4}{3} = \frac{14}{3}$ and B = $6\left(-\frac{4}{3}\right) = -8$. 6A – B = 28 – (-8) = 36.

2. **$-\frac{1}{65}$** $A = \begin{bmatrix} 16 & 23 \\ -2 & -11 \end{bmatrix}$. $A^{-1} = \frac{1}{\det A}\begin{bmatrix} -11 & -23 \\ 2 & 16 \end{bmatrix}$. $\det A = 16(-11) - (-2)(23) = -130$. So, c = $\frac{2}{-130} = -\frac{1}{65}$.

3. **2** Area = $\frac{1}{2}aP$, so $K = \frac{1}{2}aK$, and $a = 2$.

4. **$\frac{\sqrt{11}}{1331}$** Change of base results in $\left(9^{\frac{1}{2}}\right)^{\left(\log_7 \frac{1}{49}\right)(\log_4 32)(\log_9 11)} = 9^{\frac{1}{2}(-2)\left(\frac{5}{2}\right)(\log_9 11)} = 9^{\log_9 11^{\frac{-5}{2}}} = 11^{-5/2}$. This is equal to $\frac{1}{11^{5/2}} = \frac{1}{121\sqrt{11}} = \frac{\sqrt{11}}{1331}$.

5. **3π** Since $\sin 2x = 2\sin x\cos x$, rewrite as $4\sin x\cos x - 4\sin x + 2\cos x - 2 = 0$. Factor by grouping: $4\sin x(\cos x - 1) + 2(\cos x - 1) = 0$. This gives $(\cos x - 1)(4\sin x + 2) = 0$, so $\cos x = 1$ or $\sin x = -\frac{1}{2}$. This occurs for $x = 0$, $\frac{7\pi}{6}$, and $\frac{11\pi}{6}$. Sum = $\frac{18\pi}{6} = 3\pi$.

6. **98** $x(a) + y(a) = 2a - 1 + a^2 + 1 = a^2 + 2a = 9800$. Complete the square: $(a+1)^2 = 9801$, so $a + 1 = 99$, and $a = 98$.

7. **$-\frac{4}{3}$** A = 2a = $2\sqrt{7}$. B = 2c = $2\sqrt{7+2} = 6$. C = $\frac{\sqrt{2}}{\sqrt{7}} \cdot \frac{-\sqrt{2}}{\sqrt{7}} = \frac{-2}{7}$. So, $\frac{CA^2}{B} = \frac{\frac{-2}{7} \cdot \left(2\sqrt{7}\right)^2}{6} = -\frac{4}{3}$.

8. **120** $A=\lim_{x\to 5}\frac{x^2-25}{x-5}=\lim_{x\to 5}\frac{(x+5)(x-5)}{x-5}=\lim_{x\to 5}(x+5)=10$.

$B=\lim_{x\to 2}\frac{x^3-8}{x-2}=\lim_{x\to 2}\frac{(x-2)(x^2+2x+4)}{x-2}=\lim_{x\to 2}(x^2+2x+4)=12$. AB = 10(12) = 120.

9. **-14** **u** x **v** = $\begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 2 & 7 & 1 \end{vmatrix}=i+8j+21k-2k-28i-3j=-27i+5j+19k$.

$A+2^B-C=-27+2^5-19=-14$.

10. $\mathbf{\frac{349}{80}}$ This is two series alternating terms. The first series is $2+\frac{4}{7}+\frac{8}{49}+...=\frac{2}{1-\frac{2}{7}}=\frac{14}{5}$. The other series is $\frac{6}{5}+\frac{7}{25}+\frac{8}{125}+...=\left(\frac{6}{5}+\frac{6}{25}+\frac{6}{125}+...\right)+\left(\frac{1}{25}+\frac{1}{125}+...\right)+\left(\frac{1}{125}+...\right)+...$

Split into two parts. The sum $\frac{6}{5}+\frac{6}{25}+\frac{6}{125}+...=\frac{\frac{6}{5}}{1-\frac{1}{5}}=\frac{3}{2}$. The rest of the series is a sum of an infinite number of geometric series, which is $\frac{\left(\frac{1}{25}+\frac{1}{125}+...\right)}{1-\frac{1}{5}}=\frac{\frac{1}{20}}{\frac{4}{5}}=\frac{1}{16}$. Total sum is $\frac{14}{5}+\frac{3}{2}+\frac{1}{16}=\frac{224+120+5}{50}=\frac{349}{80}$.

11. **1941** $f(1)=3-12+9-7+2008=2001$. $f'(x)=15x^4-36x^2+18x-7$, and $f'(1)=15-36+18-7=-10$. $f''(x)=60x^3-72x+18$, and $f''(1)=60-72+18=6$. So, $2001+(-10)(6)=1941$.

12. $\mathbf{\frac{1}{9}}$ The possibilities for the red die are 1, 2, 2, 3, 4, 4, 5, 6, 6. For the blue they are 1, 2, 3, 4, 5, 6, 7, 8. A sum of 10 occurs with 2-8, 2-8, 3-7, 4-6, 4-6, 5-5, 6-4, and 6-4. This is 8 possibilities out of (9)(8) = 72 total combinations, so the probability is $\frac{8}{72}=\frac{1}{9}$.