Solutions Mu Area and Volume Nationals 2008

1. First find the intersections between $y = x^3 + x + 8$ and $y = 3x^2 + 7x$ or $x^3 + x + 8 = 3x^2 + 7x$ which solves $(x+2)(x-1)(x-4) = 0$ so Area =

$$\int_{-2}^{4} \left|(x^3 + x + 8) - (3x^2 + 7x)\right| dx = \int_{-2}^{1} \left(x^3 - 3x^2 - 6x + 8\right) dx + \int_{1}^{4} \left(x^3 - 3x^2 - 6x + 8\right) dx = \boxed{\frac{81}{2}}$$ **C**.

2. this needs the circumradius, which for a right triangle is half the hypotenuse. Area = $\pi\left(\frac{5}{2}\right)^2 = \boxed{\frac{25\pi}{4}}$ **C**.

3. a function for the area of this rectangle in terms of x is $A(x) = 2x \cdot \left(\left(4 - x^2\right) - \left(x^2 - 4\right)\right) = -4x^3 + 16x$. To maximize, differentiate and set to zero: $A'(x) = -12x^2 + 16 = 0$. $x = \frac{4}{\sqrt{3}}$ $A\left(4/\sqrt{3}\right) = \boxed{\frac{64\sqrt{3}}{9}}$ **D**.

4. Area $A(d) = \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{d}{2}\right)^2 - \frac{1}{2}\pi\left(\frac{1-d}{2}\right)^2$, where d is the diameter of the smallest circle; max $A'(d) = \frac{\pi}{8}\left(2 - 4d\right) = 0$, $d = \frac{1}{2}$ $A(.5) = \boxed{\frac{\pi}{16}}$ **E.**

5. This shape is a rhombus, so Area = .5($d_1 \cdot d_2$) = $\boxed{48}$ **D**.

6. This tetrahedron has edges length $3\sqrt{2}$ so the Volume is $V = \frac{\left(3\sqrt{2}\right)^3\sqrt{2}}{12} = \boxed{9}$ **B**.

7. $V = \int_1^{\infty} \frac{\pi}{x^2} dx = \lim_{h\to\infty} \int_1^{h} \frac{\pi}{x^2} dx = 0 + \pi = \boxed{\pi}$ **B**.

8. by definition $\boxed{13}$ **C.**

9. $V = \int_0^{\pi} \frac{\left(2\sin(x)\right)^2\sqrt{3}}{4} dx = \frac{\sqrt{3}}{2}\int_0^{\pi} \left(1 + \cos(2x)\right) dx = \boxed{\frac{\pi\sqrt{3}}{2}}$ **B**.

10. 20-21-29 is the triangle $\boxed{29}$ **D**.

11. $4\pi r^2 = 400\pi$ then $r = 10$ so $V = \frac{4\pi(10)^3}{3} = \boxed{\frac{4000\pi}{3}}$ **A**.

12. First the ellipse area: $A = (a \cdot b)\pi$ then torus median radius or distance from (5, 8) to 3x + 4y = -7.
$d = \frac{|aX + bY + C|}{\sqrt{a^2 + b^2}} = \frac{|3(5) + 4(8) - 7|}{\sqrt{3^2 + 4^2}} = 8$ $V = 2\pi AR = 2\pi\left(4 \cdot 2\pi\right) \cdot 8 = \boxed{128\pi^2}$ **E**.

13. $\left(\frac{80}{3 \cdot 4}\right) \cdot \left(40 + 4 \cdot 45 + 2 \cdot 50 + 4 \cdot 45 + 35\right) = \boxed{\frac{10700}{3}}$ **A**.

14. Since there are 4 pedals in this rose, integrate on interval $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

so $A = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2}\left(4\cos(2\theta)\right)^2 d\theta = 8\int_0^{\frac{\pi}{4}} \left(1 + \cos(4\theta)\right) d\theta = 8\left(\frac{\pi}{4} + 0\right) = \boxed{2\pi}$ **B**.

15. The intersection points are (0,1), (4, 1), (2, 3). Area = $\frac{1}{2} \cdot \begin{vmatrix} 0 & 1 & 1 \\ 4 & 1 & 1 \\ 1 & 4 & 1 \end{vmatrix} = \boxed{6}$ **B**.

16. $V=\pi\int_0^1\left(\sqrt{4y}\right)^2 dy=4\pi\int_0^1 y\,dy=\boxed{2\pi}$ **C**.

17. solve for x: $x^2+\dfrac{x^2\sqrt{3}}{4}=3000\quad x=\boxed{20\sqrt{\dfrac{30}{13}\left(4-\sqrt{3}\right)}}$ **E**.

18. let x = CB, and θ = angle I in radians. and so $A=\dfrac{1}{2}\cdot IC\cdot BC=\dfrac{1}{2}\cdot 6\sqrt{2}\cdot x$ but $x=6\sqrt{2}\cdot\tan(\theta)$ so $A=36\tan(\theta)$. So $A'\left(\dfrac{\pi}{3}\right)=36\sec^2\left(\dfrac{\pi}{3}\right)\cdot 1=\boxed{144}$ **A**.

19. $\pi\cdot r\cdot l=575\pi$ and $\pi\cdot r^2=529\pi$ so $r=23$ and $l=25$. Thus $h=\sqrt{25^2-23^2}=4\sqrt{6}$ so $A=\dfrac{4\sqrt{6}\cdot\pi\cdot 23^2}{3}=\boxed{\dfrac{2116\pi\sqrt{6}}{3}}$ **D**.

20. C – A = (-4, -1, -1) B – A = (1, -7, 1) So $Area=\left\|\begin{vmatrix} i & j & k\\ -4 & -1 & -1\\ 1 & -7 & 1\end{vmatrix}\right\|=\sqrt{(-8)^2+(3)^2+(29)^2}=\boxed{\sqrt{914}}$ **B**.

21. $Vol=\dfrac{1}{6}\left|(1,3,4)\cdot\left|\begin{matrix} i & j & k\\ 2 & -4 & 5\\ -3 & 2 & 2\end{matrix}\right|\right|=\dfrac{1}{6}\left|1\cdot -18+3\cdot -19+4\cdot -8\right|=\boxed{\dfrac{107}{6}}$ **E**.

22. $\pi\int_0^3\dfrac{2x}{x^2+4}dx=\ln\left|x^2+4\right|\Big|_{x=0}^{x=3}=\boxed{\pi\ln\left(\dfrac{13}{4}\right)}$ **A**.

23. $f(x)=\int_c^x\sin(t)dt$ so $f'(x)=\sin(x)$ this means that *f(x)* is maximized at $x=k\pi$ whenever $k\in\mathbb{Z}$. This means that the maximum value of *f(x)* depends on *c*. Hence Not enough information **E.**

24. Radius of sphere is easily found to be 3. Let R = radius of Cone, h = height of cone. Similar triangles shows $\dfrac{R}{h}=\dfrac{3}{\sqrt{(h-3)^2-3^2}}$, so solve that for *R*, and plug into cone volume formula: $\dfrac{1}{3}\pi\left(\dfrac{3h}{\sqrt{(h^2-6h)}}\right)^2 h$.
Differentiate, get *h* = 12, plug into first equation and get $R=3\sqrt{2}$. Finally, cone volume is $\boxed{72\pi}$ **C**.

25. 3-4-5, 5-12-13, perimeter sum = 12 + 30 = $\boxed{42}$ **B**.

26. first find the height of the pyramid $h=\dfrac{3\cdot V}{A}=\dfrac{3\cdot 48}{64}=\dfrac{9}{4}$ then use that to find the lateral height $l=\sqrt{\left(\dfrac{9}{4}\right)^2+4^2}=\sqrt{\dfrac{81+256}{16}}=\dfrac{\sqrt{337}}{4}$ then $LA=\dfrac{1}{2}\cdot 8\cdot\dfrac{\sqrt{337}}{4}\cdot 4=\boxed{4\sqrt{337}}$ **B**.

27. $Area=\dfrac{1}{2}\begin{vmatrix} 2 & 4 & 1\\ 3 & 7 & 1\\ -2 & 8 & 1\end{vmatrix}=\boxed{8}$ **B**.

28. $y=\dfrac{1}{x}\rightarrow x=\dfrac{1}{y}$ so $2\pi\int_{1/9}^{1/2} y\left(\dfrac{1}{y}-2\right)dy=\boxed{2\pi\int_{1/9}^{1/2}(1-2y)dy}$ **D**.

29. $2\pi\int_{1/9}^{1/2}(1-2y)dy=2\pi\left[\tfrac{1}{2}-\left(\tfrac{1}{2}\right)^2-\tfrac{1}{9}+\left(\tfrac{1}{9}\right)^2\right]=\boxed{\dfrac{49\pi}{162}}$ **D**.

30. The trapezoid has to have bases length 3 and 6, with height 4. The resulting area is $\boxed{18}$ **D**.