Calculus Hustle Nationals 2008 solutions

1. $\boxed{(1,2)\cup(3,\infty)}$ Speed will be increasing where velocity and acceleration or either both positive or both negative. $v(t)=3t^2-12t+9$ velocity is positive on $(0,1)\cup(3,\infty)$ and decreasing on $(1,3)$. $a(t)=6t-12$ and is positive on $(2,\infty)$ and negative on $(0,2)$. So velocity and acceleration are both positive when $t>3$ and both negative when $1<t<2$.

2. $\boxed{-\frac{5}{2}}$ $\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{3-5i}{n^2}\right) = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left(\frac{3}{n}-\frac{5i}{n}\right)=\int_0^1(-5x)\,dx=-\frac{5}{2}$

3. $\boxed{\frac{1}{2}\left(x^2+\ln\left|x^2-1\right|\right)+C}$ $\int\frac{x^3}{x^2-1}dx=\int\frac{x^2\cdot xdx}{x^2-1}$ $x^2-1=u\to x^2=u+1\to xdx=\frac{1}{2}du$

$\frac{1}{2}\int\frac{u+1}{u}du=\frac{1}{2}\int\left[1+\frac{1}{u}\right]du=\frac{1}{2}\left[u+\ln|u|+c\right]=\frac{1}{2}\left[x^2-1+\ln\left|x^2-1\right|+C\right]=\frac{1}{2}\left[x^2+\ln\left|x^2-1\right|\right]+C$

4. $\boxed{2^{x^3}3x^2\ln 2}$ $y=2^{x^3}\to\ln y=x^3\ln 2$ $\frac{1}{y}y'=3x^2\ln 2\to y'=y\cdot 3x^2\ln 2=2^{x^3}3x^2\ln 2$

5. $\boxed{\frac{1}{e^{\pi}}}$ $\frac{dy}{dx}=\frac{\frac{dy}{dt}}{\frac{dx}{dt}}=\frac{-\pi\sin(\pi t)}{\pi e^{\pi t}}=-\frac{\sin(\pi t)}{e^{\pi t}}$.

$$\frac{d^2y}{dx^2}=\frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}}=\frac{\frac{-\cancel{\pi}\cos(\pi t)\cancel{e^{\pi t}}+\cancel{\pi}\sin(\pi t)\cancel{e^{\pi t}}}{e^{2\pi t}}}{\cancel{\pi}\cancel{e^{\pi t}}}=\frac{\sin(\pi t)-\cos(\pi t)}{e^{2\pi t}}\text{ when t = .5}$$

$$\frac{d^2y}{dx^2}=\frac{1}{e^{\pi}}$$

6. $\boxed{\frac{2x}{e^{x^6}}}$ 2nd Fundamental Theorem: $\frac{d}{dx}\left[\int_{-x^2}^{1}e^{t^3}dt\right]=0-e^{-x^6}\cdot-2x=\frac{2x}{e^{x^6}}$

7. $\boxed{\frac{99}{14}}$ $y-y_0=\frac{dy}{dx}(x-x_0)$ $y-7=\frac{1}{2\sqrt{49}}(50-49)\to y=\frac{1}{14}+7=\frac{99}{14}$

8. $\boxed{\frac{\pi}{6}}$ $\int_0^{\ln\sqrt{3}}\frac{e^x}{\sqrt{4-e^{2x}}}dx=\int_0^{\ln\sqrt{3}}\frac{e^x}{\sqrt{2^2-\left(e^x\right)^2}}=\arcsin\left(\frac{e^x}{2}\right)\Big|_0^{\ln\sqrt{3}}=\arcsin\left(\frac{\sqrt{3}}{2}\right)-\arcsin\left(\frac{1}{2}\right)=\frac{\pi}{6}$

9. $\boxed{84\pi^2}$ Use Pappus. Equation of ellipse is $\frac{(x-2)^2}{4}+\frac{(y+7)^2}{9}=1$. Area of ellipse is $ab\pi = 6\pi$. Distance from center to axis of revolution is 7. $V = 2\pi rA = 2\pi \cdot 7 \cdot 6\pi = 84\pi^2$

10. $\boxed{e}$ $\lim_{x\to 0}\left(1+\frac{1}{y}\right)^y$ definition of e

11. $\boxed{-5}$ The y-coordinates when x =1 are:

$y+3y^2=2 \to 3y^2+y-2=0 \to (3y-2)(y+1)=0 \therefore y=-1,\frac{2}{3}$. Since we want IV quadrant we use the point (1,-1) . Differentiating implicitly we get

$2xy+x^2\frac{dy}{dx}+3y^2+6xy\frac{dy}{dx}=0 \;\to \frac{dy}{dx}=\frac{-\left(2xy+3y^2\right)}{x^2+6xy} \to m=\frac{1}{5}$ So norm will be -5.

12. $\boxed{2}$ $V=s^3 \to \frac{dV}{dt}=3s^2\frac{ds}{dt}$ When the volume of the cube is 8, the side is 2 and

$SA=6s^2 \to \frac{dSA}{dt}=12s\frac{ds}{dt} \to 4=24\frac{ds}{dt} \therefore \frac{ds}{dt}=\frac{1}{6}$ so $\frac{dV}{dt}=3s^2\frac{ds}{dt}=3(2)^2\frac{1}{6}=2$

13. $\boxed{\frac{1}{5}}$ $y=\sqrt{x+3\sqrt{x+3\sqrt{x+3...}}} \to y^2=x+3y \to 2y\frac{dy}{dx}=1+3\frac{dy}{dx} \to \frac{dy}{dx}=\frac{1}{2y-3}$

$y=\sqrt{4+3\sqrt{4+3\sqrt{4+3...}}} \to y^2=4+3y \to y^2-3y-4=0 \;\; (y-4)(y+1)=0 \to y=-1,4$

The sum cannot be negative so y = 4. $\frac{dy}{dx}=\frac{1}{2y-3}=\frac{1}{8-3}=\frac{1}{5}$

14. $\boxed{-2\le x<6}$ $\lim_{x\to\infty}\frac{\frac{(x-2)^{n+2}}{(n+2)4^{n+2}}}{\frac{(x-2)^{n+1}}{(n+1)4^{n+1}}} = \frac{\frac{(x-2)^{\not{n+2}}}{(n+2)4^{\not{n+2}}}}{\frac{\not{(x-2)}^{\not{n+1}}}{(n+1)\not{4}^{\not{n+1}}}}=\lim_{n\to\infty}\frac{(n+1)(x-2)}{(n+2)4}=\frac{x-2}{4}$

$\left|\frac{x-2}{4}\right|<1 \to -2<x<6$ Plugging in endpoints only -2 converges: $-2\le x<6$.

15. $\boxed{12096}$ The coefficient x^5 term of $f'(x)$ will be the coefficient of the x^6 term of f(x) multiplied by 6. The coefficient of the x^6 is $2^4\cdot 1^5\cdot\frac{9!}{5!4!}=2016$. Multiplying by 6 per the power rule we get 12096.

16. $\boxed{y=2x^2e^x}$ $\frac{dy}{dx}=\frac{y(x+2)}{x}\to\int\frac{dy}{y}=\int\frac{(x+2)\,dx}{x}\to\ln y=x+2\ln x+C$

$\ln(2e)=1+2\ln 1+C\to\ln 2+1=1+C\to C=\ln 2$ $\ln y=x+2\ln x+\ln 2\to e^{x+\ln(2x^2)}=y$

$y=2x^2e^x$

17. $\boxed{1/4}$ $g'(x)=\frac{1}{f'(g(x))}$ $\tan x=\sqrt{3}, x=\frac{\pi}{3}$ $\frac{d}{dx}[\tan x]=\sec^2 x\to\sec^2(\pi/3)=4$ so slope of inverse is $1/4$.

18. $\boxed{55}$ $f'(x)=15x^4-15x^2=15x^2(x+1)(x-1)=0$ $x=\pm1,0$, check bounds, $f(2)$ yields a value of 55

19. $\boxed{\frac{\pi}{3}}$ $r=2\cos(3\theta)=0\to\theta=\frac{\pi}{6},\frac{\pi}{2}$ $A=\frac{1}{2}\int_{\pi/6}^{\pi/2}(2\cos(3\theta))^2\,d\theta=\frac{1}{2}\int_{\pi/6}^{\pi/2}4\cos^2(3\theta)\,d\theta$

$$=2\int_{\pi/6}^{\pi/2}\frac{1+\cos(6\theta)}{2}=\int_{\pi/6}^{\pi/2}1+\cos(6\theta)=\theta+\frac{\sin 6\theta}{6}\Big|_{\pi/6}^{\pi/2}=\frac{\pi}{3}$$

20. $\boxed{\frac{72\pi}{5}}$ Set the functions equal to each other and solve for x. The left bound of the region being revolved is x = 0 and the right bound is x = 3. Use washer

$$\pi\int_0^3\left[(x+3)^2-(x^2)^2\right]dx=\pi\int_0^3(x^2+6x+9-x^4)\,dx=\frac{72\pi}{5}$$

21. $\boxed{-\frac{\pi}{5}}$ $\frac{dA}{dt}=\pi\left(3\cdot\frac{3}{5}+5\cdot\left(-\frac{2}{5}\right)\right)=-\frac{\pi}{5}$ $A=ab\pi\to\frac{dA}{dt}=\pi\left(b\frac{da}{dt}+a\frac{db}{dt}\right)$

$$\frac{dA}{dt}=\pi\left(3\cdot\frac{3}{5}+5\cdot\left(-\frac{2}{5}\right)\right)=-\frac{\pi}{5}$$

22. $\boxed{\frac{2}{x}-\cot(3x)\text{ or }\frac{2}{x}-\frac{\cos(3x)}{\sin(3x)}}$ $y=\ln\sqrt[3]{x^6\sin(3x)}=\frac{1}{3}\ln\left(x^6\sin(3x)\right)=$

$\frac{1}{3}\ln(x^6)+\frac{1}{3}\ln(\sin 3x)=2\ln x+\frac{1}{3}\ln(\sin 3x)$ $y'=\frac{2}{x}-\frac{1}{3}\cdot\frac{1}{\sin 3x}\cdot\cos 3x\cdot 3=\frac{2}{x}-\cot(3x)$

or $\frac{2}{x}-\frac{\cos(3x)}{\sin(3x)}$

23. $\boxed{1+\sqrt{21}}$ MVT for Integrals $f(c)=\dfrac{1}{b-a}\int_a^b f(x)dx$

$\dfrac{1}{7-4}\int_4^7 (x-1)^2\,dx=\dfrac{1}{9}(x-1)^3\Big|_4^7=21\ \dfrac{1}{7-4}\int_4^7 (x-1)^2\,dx=\dfrac{1}{9}(x-1)^3\Big|_4^7=21=\ 21=\ x^2-2x+1$

$x^2-2x-20=0\rightarrow x=1\pm\sqrt{21}$ Only $1+\sqrt{21}$ is on the interval.

24. $\boxed{-\dfrac{\sqrt{3}}{1440}}$ The x^6 term of the Taylor expansion will be $\dfrac{f^6\left(\pi/6\right)\left(x-\pi/6\right)^6}{6!}$ where

$f^n(x)$ denotes the nth derivative. $f^6(\cos x)=-\cos\text{ x},\ -\cos\left(\pi/6\right)=-\dfrac{\sqrt{3}}{2}$.

$\dfrac{-\sqrt{3}/2}{6!}=\dfrac{-\sqrt{3}}{1440}$.

25. $\boxed{\dfrac{9x-18}{2x^2}}$ If $f\left(\dfrac{3}{x}\right)=x^2-3x$ then $f(x)=\left(\dfrac{3}{x}\right)^2-3\left(\dfrac{3}{x}\right)=\dfrac{9}{x^2}-\dfrac{9}{x}$

$=\dfrac{9x-18/x^3}{2/x}=\dfrac{9x-18}{x^2}\ \ f'(x)=\dfrac{9x-18}{x^3}$

$\dfrac{d\left[f(x)\right]}{d(2\ln x)}=\dfrac{9x-18/x^3}{2/x}=\dfrac{9x-18}{2x^2}$