Let $\mathbb{N} = \{1, 2, 3, \cdots\}$, $\mathbb{Z}$ = the set of integers, $\mathbb{Q}$ = the set of rational numbers, $\mathbb{R}$ = the set of real numbers, and $\mathbb{C}$ = the set of complex numbers. We will let $i = \sqrt{-1}$. Enjoy the test!

1. Which of the following functions are odd?

   I. $3x^3|x| - 2x$
   II. $\cosh(x)$
   III. $\ln(x + \sqrt{x^2+1})$

   (A) I only (B) II only (C) I, III (D) I, II (E) NOTA

2. Define $f : \mathbb{Q} \to \mathbb{Z}$ where $f\left(\frac{p}{q}\right) = p$, where $p, q \in \mathbb{Z}$ and $q \neq 0$. Why is f not a function?

   I. f is not surjective (onto).
   II. f is not injective (1-1).
   III. f is not well-defined (every input has exactly one output).

   (A) I only (B) II only (C) I, III (D) I, II (E) NOTA

3. Which of the following functions counts the number of divisors of an arbitrary $n \in \mathbb{N}$?

   (A) $\gamma(n)$ (B) $\sigma(n)$ (C) $\varphi(n)$ (D) $\pi(n)$ (E) NOTA

4. Which of the following are true about $f(x) = \begin{cases} -x-2 & x \leq -2 \\ -\frac{2}{5}x + \frac{16}{5} & -2 < x < 3 \\ x^2 - 6x + 11 & x > 3 \end{cases}$

   I. The discontinuity at $x = -2$ is removable.
   II. $\lim\limits_{x \to -2^+} = f(-2)$.
   III. $\lim\limits_{x \to 3} f(x)$ exists.

   (A) I only (B) III only (C) I, II (D) I, II, III (E) NOTA

5. Consider $f : x \mapsto x^2 - 4x$. Find the range of f.

   (A) $[2, \infty)$ (B) $[-4, \infty)$ (C) $(-\infty, 2]$ (D) $(-\infty, -4]$ (E) NOTA

6. Let $Z(f)$ denote all the *real* zeros of a function f. If $f(x) = 12x^3 - 8x^2 - 27x + 18$, find the product of all members of $Z(|f|)$.

   (A) $-\frac{27}{8}$ (B) $-\frac{3}{2}$ (C) $\frac{27}{8}$ (D) $\frac{9}{4}$ (E) NOTA

For the following three questions, let $g(x) = \dfrac{x^3 - 3x^2 - 9x + 27}{x^2 + 2x - 15}$

7. If (x, y) represents the coordinates of the hole in the graph of g, find $x + y$:

(A) 3 (B) 0 (C) (D) There is no hole

(E) NOTA

8. Indicate the number of zeros of g:

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

9. Find the x-intercept of the slant asymptote:

(A) $(-5, 0)$ (B) $(5, 0)$ (C) $(0, 0)$ (D) $(0, -5)$ (E) NOTA

10. Define $f(x) = \alpha x^5 + \beta x^4 + \gamma x^3 + \delta x^2 + \epsilon x + \zeta$, such that f maps the complex to the complex, and $\alpha, \beta, \gamma, \delta, \epsilon, \zeta \in \mathbb{C}$. Provided that ω is a non-real zero of f, which of the following must be true?

I. $\overline{\omega}$ is a zero of f

II. f has no more than 3 real zeros

III. $\dfrac{\zeta}{\alpha}$ is possible root of f

(A) I only (B) III only (C) I, II (D) I, II, III (E) NOTA

For the next two problems, consider the following: Let $\ell : \{1, 2, \ldots, 12\} \to \mathbb{N}$ **be the function which maps an integer to the number of letters in its standard English name. So** $\ell(1) = 3$ **since "one" has 3 letters in it, and** $\ell(2) = 3$**. Now, let** $\mu : \{1, 2, \ldots, 12\} \to \mathbb{N}$ **be the calendar map, where** $\mu(1) = 7$ **since January is the first month, and "January" has seven letters.** $g(2)$ **is the number of letters in "February", so** $g(2) = 8$**.**

11. Find $(\ell \circ \mu)(11)$.

(A) 3 (B) 8 (C) 5 (D) Not defined (E) NOTA

12. What is the smallest value of n such that $\ell^{(k)}(12) = \ell^{(n)}(12)$ for all $k \geq n$, where $\ell^{(n)}(x)$ is ℓ composed with itself n times, so that $\ell^{(1)}(x) = \ell(x)$, $\ell^{(2)}(x) = \ell(\ell(x))$, etc.

(A) 2 (B) 3 (C) 4 (D) 5 (E) NOTA

13. Let β be a function which maps the integers base ten to the integers base eight, where $\beta(x) = 0_{eight}$ if x contains the numerals 8 or 9. If x does not contain the numerals 8 or 9, then $\beta(x)$ in base eight has the same representation as x in base ten. For example, $\beta(20) = 20_{eight}$ and $\beta(1234567) = 1234567_{eight}$, but $\beta(91) = 0_{eight}$ and $\beta(8989891234567) = 0_{eight}$. Which of the following are true?

I. β is injective

II. β is surjective

III. β is well-defined

(A) I only (B) II only (C) II, III (D) I, II, III (E) NOTA

14. Let $f(x) = x^2 + 4x - 7$. For how many non-positive integral values of x does $f(x)$ take on negative values?

(A) 5 (B) 6 (C) 7 (D) ∞ (E) NOTA

15. Let $g(x) = \big| |x+2| - 3 \big|$, and let a be the number of relative minima, b be the number of relative maxima, and c be the product of the zeros. Find $a + 2b - c$.

(A) -1 (B) -2 (C) 8 (D) 9 (E) NOTA

16. In the function $g(x)$ given above, what is the y-coordinate of the y-intercept?

(A) -3 (B) -1 (C) 1 (D) 3 (E) NOTA

17. Let $\psi(x)$ be a function which is both odd and even. If possible, find $\psi(-2)$.

(A) -2 (B) 0 (C) 2

(D) A function cannot be both even and odd (E) NOTA

18. Find the equation of the resulting function when reflecting $f(x) = x^3$ about the line $y = x - 2$.

(A) $\sqrt[3]{x-2} - 2$ (B) $\sqrt[3]{x-2} + 2$ (C) $\sqrt[3]{x+2} - 2$ (D) $\sqrt[3]{x+2} + 2$ (E) NOTA

19. Consider the graph of a quadratic function in the Cartesian plane, where x is considered to be a function of y. How many of the following must be true?

- The function may have more than one y-intercept.
- The function must have one or more x-intercepts.
- The horizontal line test will determine if the function is one-to-one.
- The range is all real numbers.

(A) 0 (B) 1 (C) 2 (D) 3 (E) NOTA

20. Let $\eta : \mathbb{Z} \to \mathbb{R}$ be a 1-1 function such that $\eta^{-1}(3) = 4$ and $\eta(k+1) = \frac{1}{5}\eta(k)$. Find $\sum_{i=-1}^{\infty} \eta(i)$.

(A) $\frac{375}{4}$ (B) $\frac{1875}{4}$ (C) $\frac{9375}{4}$ (D) $\frac{46875}{4}$ (E) NOTA

21. Find the range of $q(x) = \frac{2x+1}{x-3}$ if the domain is $\{x|x \neq 3\}$:

(A) $\left\{x|x \neq -\frac{1}{3}\right\}$ (B) $\left\{x|x \neq -\frac{1}{2}\right\}$ (C) $\{x|x \neq 2\}$ (D) $\{x|x \neq 3\}$ (E) NOTA

22. The function $f : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}$, has the property that for each x in its domain D, then $\frac{1}{x}$ is also in D, and $f(x) + f\left(\frac{1}{x}\right) = x$. Find the largest possible domain, D.

(A) $\{x : x \neq 0\}$ (B) $\{x : x > 0\}$ (C) $\{x : x < 0\}$ (D) $\{-1, 1\}$ (E) NOTA

23. Suppose that $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ where X and Y are non-empty subsets of $\mathbb{R}$. Suppose also that for all $x \in X$, the compositions are defined and $(f \circ g)(x) = (g \circ f)(x)$. How many of the following five statements must be true?

- $X \subseteq Y$
- $Y \subseteq X$
- f and g are inverses
- both f and g are one-to-one
- both f and g are continuous

(A) 2 (B) 3 (C) 4 (D) 5 (E) NOTA

24. Define $f(x) = \big||x-3|-4\big|$, and note that f can be written as a piecewise function with 4 branches. Given that the equation of each branch of f can be written in the form $y = ax + b$, what is $a + b$ when $x = 2$?

(A) -2 (B) 2 (C) 6 (D) 8 (E) NOTA

25. Let $f(x) = mx + b$ be a linear function. For which of the following situations must f pass through infinitely many lattice points? (A lattice point is an ordered pair in the $xy-$plane with integer values).

I. m is rational, not equal to zero
II. m is irrational
III. $m = 0$

(A) I only (B) III only (C) I, III (D) I, II, III (E) NOTA

26. Let $f(x) = x, g(x) = x^2, h(x) = x^3$, and let all be defined for all real numbers. Which of the following is never true?

(A) $f(x) < h(x) < g(x)$ (B) $g(x) < h(x) < f(x)$ (C) $h(x) < f(x) < g(x)$
(D) $h(x) < g(x) < f(x)$ (E) NOTA

27. Let $f(x)$ be a function. Translate the function up 2 units, and to the right 3 units. Then, reflect it over the y-axis, and compress it vertically by a factor of $\frac{1}{2}$. What is the equation of this function?

(A) $\frac{1}{2}f(-x+3)+1$ (B) $\frac{1}{2}f(-x+3)+2$ (C) $\frac{1}{2}f(-x-3)+2$ (D) $\frac{1}{2}f(-x-3)+1$
(E) NOTA

28. Find $\lim_{x \to 4} f(x)$ when $f(x) = \dfrac{2x-8}{2x-\sqrt{4x^2-3x+12}}$.

(A) 0 (B) $\frac{2}{3}$ (C) $\frac{16}{3}$ (D) $\frac{32}{3}$ (E) NOTA

29. Let $f(z) = e^{2z}$ be a complex-valued function. Find the period of f.

(A) π (B) $i\pi$ (C) 2π (D) $i2\pi$ (E) NOTA

30. Find the axis of symmetry of $f(x) = -12x + 5 - 2x^2$.

(A) $x = -3$ (B) $x = -\frac{1}{12}$ (C) $x = \frac{5}{24}$ (D) $x = 3$ (E) NOTA

TB1 Find $\lim_{x\to 0^+} g(x)$ when $g(x) = \lfloor -x \rfloor$, where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

TB2 Find k so that the following function is continuous: $f(x) = \begin{cases} x^2 - 4x + 7 & x \leq 2 \\ \frac{k}{3}x + 11 & x > 2 \end{cases}$

TB3 Find the range of $h(x) = \arctan(-e^x)$ if we take the domain to be all positive real numbers.