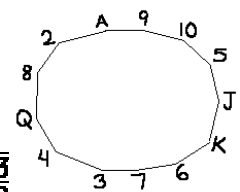
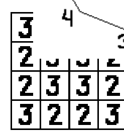
Solutions

1. **B.** Since ${}_nC_r$ can only take values of *n* and *r* such that $n \geq r \geq 0$, factor each term to determine which values can be allowed. The *n* term factors as $-(x + 1)(x - 9)$ and the *r* term factors as $-x(x - 6)$. Because of the domain condition, $0 \leq x \leq 6$ and so our 7 choices are $\binom{9}{0},\binom{16}{5},\binom{21}{8},\binom{24}{9},\binom{25}{8},\binom{24}{5},\binom{21}{0}$ The two largest choices are $\binom{24}{9}$ and $\binom{25}{8}$. Verify $\binom{24}{9} > \binom{25}{8}$ by cross multiplying. $\frac{24!}{9!15!} > \frac{25!}{8!17!} \rightarrow \frac{8!17!}{9!15!} > 25 \rightarrow \frac{272}{9} > 25$

2. **A**. Since the reordering cycles all 13 cards, one could execute the double reordering 6 more times in order to get back to the first reordering. Also helpful is a 13-gon diagram in which the single reordering is given by moving to the next vertex. After two reorderings, A → 10, 10 → J, J → 6 and so on. The third reordering then takes A → 5, 5 → 6, 6 → 4 and so on.



3. **A.** The diagram to the right shows the weights for all the cells. There are only two distinct weights used.



4. **B**. The average value of n^3 on the interval [0, 4] is given by $\frac{1}{4}\int_0^4 x^3 dx = 16$.

5. **D**. Because the telephone number is divisible by both 2 and 5, it must end in 0. Also, because it's divisible by 3, the sum of the remaining 6 digits is congruent to 0 (mod 3).
1, 4, 7 are ≡ 1 (mod 3). 2, 5, 8 are ≡ 2 (mod 3). 3, 6, 9 are ≡ 0 (mod 3). One can take the three digits from a pair of groups. There are 3 ways to do this and 6! ways to rearrange the chosen digits. The only other way to get 0 (mod 3) is to take two numbers from each of the three groups. There are (3C2)(3C2)(3C2) = 27 ways to do this and the 6! ways to rearrange them, making for a total of 30 times 6! of different numbers.

6. **B**. Start by examining the first terms of this sequence. With 1 line, 2 regions can be made. With 2 lines, 4 regions can be made. With 3 lines, 7 regions can be made. The sequence is as follows: 2, 4, 7, 11, 16… These numbers increase as the triangular numbers do but have 1 added to each of them. For 17 lines, we want the 17th triangular number plus 1. 17(18)/2 + 1 = 153 + 1 = 154

7. **B**. Because the derivative is at *x* = 0, we need to consider terms that are x^{23}. Our multinomial coefficient in the expansion is $\binom{5}{a,b,c}$. We need $7a + 3b + 2c = 23$ due to the degree and $a + b + c = 5$. We can subtract the second equation from the first twice to yield 5a + b = 13. Use this condition to determine that the only triple that will satisfy the first two restrictions is (2, 3, 0). This means our multinomial coefficient is 10 and the x^{23} term is $10(2x^7)^2(-x^3)^3 = -40x^{23}$ Differentiating this 23 times yields –40 times 23!

8. **B**. Treat Dylan and Amanda as one person, leaving 5! ways to rearrange them. Then treat Dylan and Courtney as one person, leaving another 5! ways arrange them. However, some of the first 5! ways are counted again in the second 5! ways. So we need to subtract the 4! that Courtney, Dylan, and Amanda all stand as one person. 5! + 5! – 4! = 216.

9. **C.** There are a total of 2^7 sets and half of them contain 2, narrowing our pool to 2^6 Now we have to subtract the number of subsets that have the 2 and both L and M. The possible subsets of this nature will take the form

{_ 2 _ _ _ L M}. Each blank will either be filled or not be filled. Two choices for each blank with four blanks gives 16 possible subsets of this nature. Subtract this from the pool of 64 to give 48.

10. **B.** First evaluate $\frac{d}{dx}\left(\frac{x^2}{2}+2\right)^{12} = 12x\left(\frac{x^2}{2}+2\right)^{11}$. We want to choose the value for *k* such that $x\left(x^2\right)^k = x^{2k+1} = x^{15}$. Therefore *k* = 7. $12x\frac{11!}{7!4!}\left(\frac{x^2}{2}\right)^7(2)^4 = 495x^{15}$

11. **C.** The largest value for Q is 15. The fact that $|B \cap D| = 7$ and $|C \cap B| = 4$ do not detract or add to the largest value for $|A \cap C \cap D|$. All of C can be contained within A and D, causing Q = 15. Then, none of the restrictions cause A to share any of its elements with any other set, so it could be disjoint from the other sets. Thus $|A \cap B \cap D|$ = 0 = Z.

12. **B**.

13. **B**. x^{2x} is a perfect cube anytime that *x* is a multiple of 3. There are 333 multiples of 3 on 0 < *x* < 1000. However. There are also 6 perfect cubes *x* on (0, 1000) that aren't multiples of 3. Namely: 1, 8, 64, 125, 343, 512. If *x* is any one of these, then x^{2x} will also be a perfect cube. So there are 339 values of *x*.

14. **B**. $f'(x) = \prod_{n=1}^{101}(n-x)^n = (1-x)(2-x)^2...(100-x)^{100}(101-x)^{101}$. This product produces a critical number line <++(1)--(2)--(3)++(4)++(5)--…--(99)++(100)++(101)-->
The derivative switches from positive to negative (producing maxima) at 1, 5, 9, 13.. 101. There are 101 = 1 + 4(*n* – 1) → *n* = 26 maxima.

15. **C**. All scores from 0 to 138 are possible. However, the scores 139, 143, 144, 147, 148, and 149 are not possible. 140, 141, 142, 145, 146 and 150 are the other possible scores. This makes 145 possible scores. Two people must be placed into each of these, making 290, so a 291st person will put a 3rd person in a particular score.

16. **E**

17. **D.** There are 8!/(2!2!) ways to rearrange the letters in ACCISMUS.

18. **B.** Since the sum is fixed, deciding the first number is all that matters. In the first number, the first digit can be 1, 2, or 3 (it cannot be 4 because then the second number would only be 2 digits). The second digit can be 0, 1, 2, or 3. The third digit can be 0, 1, or 2. This is (3)(4)(3) = 36 choices.

19. **A.** The last page is labeled $179_{10} = 263_8$. The first 7 pages have 1 digit. Then the two digit pages contain {10, 11, … 17, 20, 21…76, 77} of which there are 56 pages. The three digit pages starting with 1 contain {100, 101…107, 110, 111, … 176, 177} of which there are 64. The three digit pages starting with 2 contain {200, 201…263} of which there are 52. 7(1) + 56(2) + 116(3) = 467

20. **C**. The number of solutions (x_1, x_2, x_3, x_4) that satisfy $x_1 + x_2 + x_3 + x_4 \le 7$ is equal to the number of solutions $(x_1, x_2, x_3, x_4, x_5)$ that satisfy $x_1 + x_2 + x_3 + x_4 + x_5 = 7$ because we can treat "$\le$ 7" as "= 7 – x_5" where x_5 is a non-negative integer. $x_1 + x_2 + x_3 + x_4 + x_5 = 7$ has 11C4 = 330 solutions.

21. **A**. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$ Since the sequence starts with *n* = 1 and we're taking reciprocals, the sum is ½ + ¼ + 1/8 + 1/16…, which converges to 1.

22. **D**. 9800 factors as $2^3 5^2 7^2$. Which means $2^3 5^2 7^2 = (2^a 5^k 7^x)(2^b 5^m 7^y)(2^c 5^n 7^z)$ as a product of three positive integers gives us three equations *a* + *b* + *c* = 3, *k* + *m* + *n* = 2,
x + *y* + *z* = 2. The number of ways to solve the three (5C3), (4C2), and (4C2) respectively, giving a total of 360 different possibilities for products.

23. **E**. Use the first condition and place Elise in first: _ _ C _ _ E. Brad finishing ahead of Courtney and Frank being in the top 3 forces Amanda to be 5th and Dylan last. There are two ways (2 total) to rearrange Brad and Frank in the top 3. Elise could not be 2nd because it would force Courtney and Amanda to be 5th, which is impossible. Place Elise 3rd and assume Frank is first with Amanda second: C _ _ E A F. There are two ways to place Dylan and Brad after that (4). Then assume Amanda is 5th: C A _ E _ F. There are two ways to place Dylan and Brad after that (6). Now suppose Frank is second. This forces Amanda to be 5th: C A _ E F _. There are two ways to place Brad and Dylan after this. 8 ways total.
24. **D**. There are a total of 2^{13} subsets. Those subsets that have only an odd number of elements constitute half of these. Therefore $x = 4096$.
25. **D**. 1029! has (1029/7) + (1029/49) + $\lfloor 1029/343 \rfloor$ = 147 + 21 + 3 = 171 powers of 7. 490! has (490/7) + (490/49) + $\lfloor 490/343 \rfloor$ = 70 + 10 + 1 = 81 powers of 7. 171 – 2(81) = 171 – 162 = 9 powers of 7 left. The largest possible value for n is 9.
26. **E**. $n^2 + 12 = n^2 - 1 + 13 = (n-1)(n+1) + 13$. If n is one less or one more than a multiple of 13, then the product $(n-1)(n+1)$ will be a multiple of 13, meaning the expression $(n-1)(n+1) + 13$ will be divisible by 13. This happens for infinitely many n.

27. **C**. Suppose the nth row has the $r-1$, r, and $r+1$ terms in the ratio 2:4:7. Then $4\binom{n}{r-1} = 2\binom{n}{r}$ and $7\binom{n}{r} = 4\binom{n}{r+1}$. Taking the first equation, we get $\frac{4n!}{(r-1)!(n-r+1)!} = \frac{2n!}{r!(n-r)!}$. Cross multiply for $4r = 2(n-r+1) \rightarrow 6r = 2n+2$.
$\frac{7n!}{r!(n-r)!} = \frac{4n!}{(r+1)!(n-r-1)!}$. Cross multiply for $7(r+1) = 4(n-r) \rightarrow 7r + 7 = 4n - 4r$.
Solving this system of two equations for n gives us that $n = 32$. The 32nd row.
28. **C.** Choose one dot from each side for three of the four sides.
$\binom{4}{1}\binom{3}{1}\binom{2}{1} + \binom{4}{1}\binom{2}{1}\binom{1}{1} + \binom{4}{1}\binom{3}{1}\binom{1}{1} + \binom{3}{1}\binom{2}{1}\binom{1}{1}$ = 24 + 8 + 12 + 6 = 50. You can also take two dots from sides that have at least that many and choose any remaining dot. $8\binom{2}{2} + 7\binom{3}{2} + 6\binom{4}{2}$ = 8 + 21 + 36 = 65.
29. **C**. The rigorous restrictions force $a = b$. If $f(x)$ was ever less than 0, there might be a clever way to choose a and b, but $f(x) > 0$ does not allow for this. c is forced to equal 9. There are 9 positive integer choices for $a = b$ that satisfy the restrictions.
30. **C**. Blue cards are elements of the set {2, 3, 5, 7}. The number of ways Jessie can NOT draw a blue card or 6 on the first draw is 5. Then on the second draw, Jessie has to draw a blue card or the 6, which gives her 5 possibilities (total 25). However, she could also draw a blue card or the 6 on her first draw in 5 ways, and then is forced to draw a blue or the 6 (if possible) on her second draw in 4 ways. 4(5) = 20 giving a total of 45 possible different ways.

Tie-breaker 1) **729.** There are 9 digits for each place making 729 possible answer.
Tie-breaker 2) **1.** Since 144 factors have only powers of 2 or 3 in them, only the factor 1 will not.
Tie-breaker 3) **20**. This triple sum is stacking the sum of the sum of the first 1, 2, 3, and 4 natural numbers into pyramidal numbers. The 4th pyramidal number is 4(5)(6)/6 = 20.