2009 Mu Sequences and Series

Solutions

Solutions to 2009 Mu Sequences and Series Test:

1) (D) $S = \dfrac{5/7}{1-7/8} = \dfrac{5/7}{1/8} = 40/7$

2) (A) This sequence is not monotone because it bounces between negative and positive numbers, and this rules out (C). All the numbers of the sequence are in the range $[-1,1]$, and this rules out (B) and (D). Since the limit of the terms is 0, this sequence is convergent.

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots$$

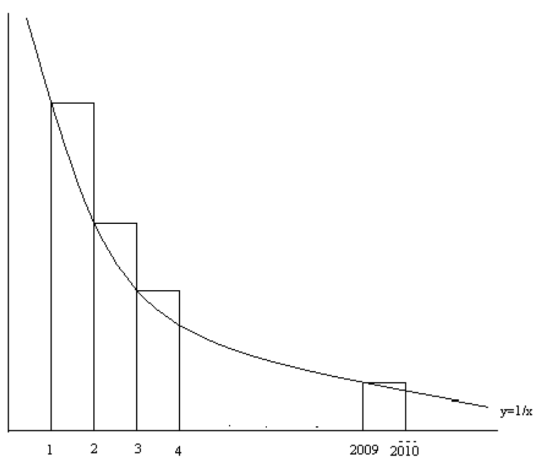
3) (B) $\frac{1}{2}\ln 2 = 0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \ldots$ and this is the desired series.

$$\frac{3}{2}\ln 2 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots$$

4) (D) The sequence of partial sums of this series is $3,0,3,0,3,0,3,0,\ldots$, which diverges. Therefore, the series diverges.

5) (D) This is the same sequence as the terms in the series for problem 3. It was given that the series is conditionally convergence in problem 3, so by the Riemann Rearrangement Theorem, there exists a rearrangement of the terms in the series that adds to any real number. So I, II, and IV are correct, and III is not.

6) (C) According to the picture, $1(1) + \frac{1}{2}(1) + \frac{1}{3}(1) + \ldots + \frac{1}{2009}(1) > \int_1^{2010} \frac{1}{x}\,dx = \ln x\Big|_1^{2010} = \ln 2010$. So $A > B$



7) (B) Direct Comparison and Limit Comparison Tests may only be used with series with positive terms only, so this rules out (C) and (D).

The Ratio Test is inconclusive since $\lim\limits_{n\to\infty}\left|\dfrac{(-1)^{n+1}\dfrac{\sqrt{n+1}}{n+2}}{(-1)^{n}\dfrac{\sqrt{n}}{n+1}}\right| = \lim\limits_{n\to\infty}\dfrac{\sqrt{n+1}}{\sqrt{n}}\cdot\dfrac{n+1}{n+2} = 1$, so this rules out (A).

^{1}x

Alternating Series Test works since $\dfrac{\sqrt{n}}{n+1} \geq \dfrac{\sqrt{n+1}}{n+2}$ (denominator increases by 1 while numerator increases by less than 1) and $\lim\limits_{n\to\infty}\dfrac{\sqrt{n}}{n+1} = 0$

9) (A) For this series, $\dfrac{4/9}{1-r} = n$ for some positive integer n. If the second term is to be as small as possible, the ratio r must be as small as possible, meaning $1-r = 4/9$, or $r = 5/9$. The second term would then be $4/9 \cdot 5/9 = 20/81$.

10) (B) The p-Series test is derived from the integral test:
$\int_n^\infty \frac{1}{x^p}\,dx = \left.\frac{x^{1-p}}{1-p}\right|_n^\infty$, which only converges if $p > 1$ (if $p = 1$, this is integrated differently)

11) (C)

$$S = \frac{2}{2}+\frac{5}{4}+\frac{10}{8}+\frac{17}{16}+\frac{26}{32}+\ldots$$
$$-\frac{1}{2}S = -\frac{2}{4}-\frac{5}{8}-\frac{10}{16}-\frac{17}{32}-\ldots$$
$$\frac{1}{2}S = \frac{2}{2}+\frac{3}{4}+\frac{5}{8}+\frac{7}{16}+\frac{9}{32}+\ldots$$
$$-\frac{1}{4}S = -\frac{2}{4}-\frac{3}{8}-\frac{5}{16}-\frac{7}{32}-\ldots$$
$$\frac{1}{4}S = \frac{2}{2}+\frac{1}{4}+\frac{2}{8}+\frac{2}{16}+\frac{2}{32}+\ldots = 1+\frac{1}{4}+\frac{2/8}{1-1/2} = \frac{7}{4}$$

So $S = 7$.

12) (D)

$$S = \frac{2}{2}+\frac{22}{4}+\frac{78}{8}+\frac{188}{16}+\frac{370}{32}+\ldots$$
$$-\frac{1}{2}S = -\frac{2}{4}-\frac{22}{8}-\frac{78}{16}-\frac{188}{32}-\ldots$$
$$\frac{1}{2}S = \frac{2}{2}+\frac{20}{4}+\frac{56}{8}+\frac{110}{16}+\frac{182}{32}+\ldots$$
$$-\frac{1}{4}S = -\frac{2}{4}-\frac{20}{8}-\frac{56}{16}-\frac{110}{32}-\ldots$$
$$\frac{1}{4}S = \frac{2}{2}+\frac{18}{4}+\frac{36}{8}+\frac{54}{16}+\frac{72}{32}+\ldots$$
$$-\frac{1}{8}S = -\frac{2}{4}-\frac{18}{8}-\frac{36}{16}-\frac{54}{32}-\ldots$$
$$\frac{1}{8}S = \frac{2}{2}+\frac{16}{4}+\frac{18}{8}+\frac{18}{16}+\frac{18}{32} = 1+4+\frac{18/8}{1-1/2} = \frac{19}{2}$$

So $S = 76$.

13) (B) $\lim\limits_{n\to\infty}\left|\dfrac{\frac{(n+2)\cdot 3^{n+2}}{(x-2)^{n+1}}}{(n+1)\cdot 3^{n+1}}\right| = \lim\limits_{n\to\infty}\dfrac{n+1}{3(n+2)}|x-2| = \dfrac{1}{3}|x-2| < 1$, so $-1 < x < 5$. For the endpoints:

$x=-1$: $\displaystyle\sum_{n=0}^{\infty}\frac{(-3)^{n+1}}{(n+1)\cdot 3^{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n+1}$, which converges by Alternating Series test.

$x=5$: $\displaystyle\sum_{n=0}^{\infty}\frac{(3)^{n+1}}{(n+1)\cdot 3^{n+1}} = \sum_{n=0}^{\infty}\frac{1}{n+1}$, which diverges because it is the harmonic series.

So the interval of convergence is $-1 \le x < 5$.

14) (A)
$$S = \sqrt{10+\sqrt{670+S}}$$
$$S^2 - 10 = \sqrt{670+S}$$
$$S^4 - 20S^2 + 100 = 670 + S$$
$$S^4 - 20S^2 - S - 570 = 0$$

By Descartes' Rule of Signs, this polynomial only has one positive root. It is easy to check that $S = 6$ works.

15) (C) $\lim\limits_{n\to\infty}\dfrac{\frac{1}{2^n}}{\frac{1}{2^n-1}} = \lim\limits_{n\to\infty}\dfrac{2^n-1}{2^n} = 1$, so the series converges by Limit Comparison Test since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{2^n}$ converges.

Since the series converges, this rules out (A) and (D), and since the series isn't alternating, this rules out (B).

16) (B) $\lim_{n\to\infty}\left|\dfrac{\dfrac{((n+1)!)^2}{(k(n+1))!}}{\dfrac{(n!)^2}{(kn)!}}\right| = \lim_{n\to\infty}\dfrac{(n+1)^2}{(kn+k)(kn+k-1)...(kn+1)} = \begin{cases}\infty, k=1\\ \dfrac{1}{4}, k=2\\ 0, k\ge 3\end{cases}$. So the series converges if $k \ge 2$.

17) (A) $\int_0^1 e^{-x^2}dx \approx \int_0^1\left(1-x^2+\dfrac{x^4}{2}-\dfrac{x^6}{6}\right)dx = \left(x-\dfrac{x^3}{3}+\dfrac{x^5}{10}-\dfrac{x^7}{42}\right)\Big|_0^1 = 1-\dfrac{1}{3}+\dfrac{1}{10}-\dfrac{1}{42}=\dfrac{26}{35}$

18) (A) The power series for $\sin^2 x$ is $\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-...\right)\left(x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-...\right)$, and the x^4 term is $2x\left(-\dfrac{x^3}{3!}\right)=-\dfrac{x^4}{3}$. So the coefficient is $-\dfrac{1}{3}$.

19) (B)

$\lim_{n\to\infty}\left|\dfrac{\dfrac{10(n+1)+3}{(n+1)\cdot 2^{n+1}}}{\dfrac{10n+3}{n\cdot 2^n}}\right| = \lim_{n\to\infty}\dfrac{10n+13}{10n+3}\cdot\dfrac{n}{n+1}\cdot\dfrac{1}{2}=\dfrac{1}{2}$, so (A) is conclusive

$\lim_{n\to\infty}\left|\dfrac{\dfrac{n+1}{2(n+1)^3+1}}{\dfrac{n}{2n^3+1}}\right| = \lim_{n\to\infty}\dfrac{(n+1)(2n^3+1)}{n(2n^3+6n^2+6n+3)}=1$, so (B) is inconclusive

$\lim_{n\to\infty}\left|\dfrac{\dfrac{(n+1)\cdot 7^{n+1}}{(n+1)!}}{\dfrac{n\cdot 7^n}{n!}}\right| = \lim_{n\to\infty}\dfrac{n+1}{n}\cdot\dfrac{n!}{(n+1)!}\cdot 7=0$, so (C) is conclusive

$\lim_{n\to\infty}\left|\dfrac{\dfrac{(-3)^{n+1}}{(n+1)\cdot 4^{n+1}}}{\dfrac{(-3)^n}{n\cdot 4^n}}\right| = \lim_{n\to\infty}\dfrac{3}{4}\cdot\dfrac{n}{n+1}=\dfrac{3}{4}$, so (D) is conclusive

20) (C) $a_n = s_n - s_{n-1}$, where s_n is the nth partial sum. So $a_n = \dfrac{n-1}{n+1}-\dfrac{n-2}{n}=\dfrac{n(n-1)-(n+1)(n-2)}{n(n+1)}=\dfrac{2}{n(n+1)}$

21) (D) $\lim_{n\to\infty}\left|\dfrac{\dfrac{(-1)^{\frac{(n+1)(n+2)}{2}}}{3^{n+1}}}{\dfrac{(-1)^{\frac{n(n+1)}{2}}}{3^n}}\right| = \dfrac{1}{3}$, so the series converges absolutely, which rules out (A) and (C)

$$S=-\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\frac{1}{81}-\frac{1}{243}-\frac{1}{729}+\frac{1}{2187}+\frac{1}{6561}-...$$

$$S=\frac{-1/3}{1-1/81}+\frac{-1/9}{1-1/81}+\frac{1/27}{1-1/81}+\frac{1/81}{1-1/81}=\left(-\frac{1}{3}-\frac{1}{9}+\frac{1}{27}+\frac{1}{81}\right)\left(\frac{81}{80}\right)=-\frac{2}{5}$$

22) (C) The sequence for $\langle a_n\rangle$ is 1, 2, 4, 8, 16, 32, 64, …
The sequence for $\langle b_n\rangle$ is 1, 2, 4, 8, 16, 32, 63, …, so the smallest n where the sequences differ is $n=7$.

23) (A) $\displaystyle\sum_{n=0}^{\infty}\frac{1}{2^{n-1}\cdot n!}=\sum_{n=0}^{\infty}\frac{2}{2^n\cdot n!}=2\sum_{n=0}^{\infty}\frac{\left(1/2\right)^n}{n!}=2e^{1/2}=2\sqrt{e}$

24) (B) This sequence is $2,\left(3/2\right)^2,\left(4/3\right)^3,\left(5/4\right)^4,...$, and the explicit form is $a_n=\left(1+\frac{1}{n}\right)^n$, and $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$.

25) (E)

26) (A)

$$f(2)=0$$

$$f^{(n)}(x)=\frac{(-1)^{n+1}(n-1)!}{x^n}\text{ for }n\ge 1$$

$$T(x)=f(2)+\sum_{n=1}^{\infty}\frac{f^{(n)}(2)}{n!}(x-2)^n$$

$$T(x)=0+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n-1)!}{2^n\cdot n!}(x-2)^n$$

$$T(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(x-2)^n}{n2^n}$$

27. (E)

28) (A) $\displaystyle\int\frac{\tan^{-1}x}{x}\,dx\approx\int\frac{x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...}{x}dx=\int\left(1-\frac{x^2}{3}+\frac{x^4}{5}-\frac{x^6}{7}+...\right)dx=C+x-\frac{x^3}{3^2}+\frac{x^5}{5^2}-\frac{x^7}{7^2}+...$

$$=C+\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)^2}$$

29) (B) $\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{2^p3^q5^r7^s}\right)-1=\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{2^p3^q5^r}\sum_{s=0}^{\infty}\frac{1}{7^s}\right)-1=\left(\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\frac{1}{2^p3^q5^r}\left(\frac{1}{1-\frac{1}{7}}\right)\right)-1=\left(\frac{7}{6}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^p3^q}\sum_{r=0}^{\infty}\frac{1}{5^r}\right)-1$

$=\left(\frac{7}{6}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{1}{2^p3^q}\left(\frac{1}{1-\frac{1}{5}}\right)\right)-1=\left(\frac{7}{6}\cdot\frac{5}{4}\sum_{p=0}^{\infty}\frac{1}{2^p}\sum_{q=0}^{\infty}\frac{1}{3^q}\right)-1=\left(\frac{7}{6}\cdot\frac{5}{4}\sum_{p=0}^{\infty}\frac{1}{2^p}\left(\frac{1}{1-\frac{1}{3}}\right)\right)-1=\left(\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}\sum_{p=0}^{\infty}\frac{1}{2^p}\right)-1$

$=\left(\frac{7}{6}\cdot\frac{5}{4}\cdot\frac{3}{2}\cdot\frac{2}{1}\right)-1=\frac{27}{8}$

30) (C) $A=\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k)!}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right)^{2k}\right)=\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k)!}\left(\frac{\pi}{4}\right)^{2k}\right)=\cos\frac{\pi}{4}=\frac{\sqrt{2}}{2}$ So $A=B$

$B=\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k+1)!}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\right)^{2k+1}\right)=\sum_{k=0}^{\infty}\left(\frac{(-1)^k}{(2k+1)!}\left(\frac{\pi}{4}\right)^{2k+1}\right)=\sin\frac{\pi}{4}=\frac{\sqrt{2}}{2}$

Tiebreakers

TB1) (Fibonacci Sequence) This is the explicit form for the sequence 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, …

TB2) (2018040.5) Proof by induction on k that $\sum_{m=1}^{k}\left(\sum_{n=1}^{k}\left(\frac{2^n}{2^m+2^n}\right)\right)=\frac{k^2}{2}$. If $k=1$,

$\sum_{m=1}^{1}\left(\sum_{n=1}^{1}\left(\frac{2^n}{2^m+2^n}\right)\right)=\sum_{m=1}^{1}\frac{2}{2^m+2}=\frac{2}{2+2}=\frac{1}{2}=\frac{1^2}{2}=\frac{k^2}{2}$. Now assuming the statement is true for k, then:

$\sum_{m=1}^{k+1}\left(\sum_{n=1}^{k+1}\left(\frac{2^n}{2^m+2^n}\right)\right)=\sum_{m=1}^{k}\left(\sum_{n=1}^{k+1}\left(\frac{2^n}{2^m+2^n}\right)\right)+\left(\sum_{n=1}^{k+1}\left(\frac{2^n}{2^{k+1}+2^n}\right)\right)=\sum_{m=1}^{k}\left(\frac{2^{k+1}}{2^m+2^{k+1}}+\sum_{n=1}^{k}\left(\frac{2^n}{2^m+2^n}\right)\right)+\sum_{n=1}^{k+1}\left(\frac{2^n}{2^{k+1}+2^n}\right)$

$=\sum_{m=1}^{k}\left(\sum_{n=1}^{k}\left(\frac{2^n}{2^m+2^n}\right)\right)+\sum_{m=1}^{k}\left(\frac{2^{k+1}}{2^m+2^{k+1}}\right)+\sum_{n=1}^{k+1}\left(\frac{2^n}{2^{k+1}+2^n}\right)=\frac{k^2}{2}+\sum_{m=1}^{k}\left(\frac{2^{k+1}}{2^m+2^{k+1}}\right)+\sum_{n=1}^{k}\left(\frac{2^n}{2^{k+1}+2^n}\right)+\frac{2^{k+1}}{2^{k+1}+2^{k+1}}$

$=\frac{k^2}{2}+\sum_{m=1}^{k}\left(\frac{2^{k+1}}{2^m+2^{k+1}}\right)+\sum_{n=1}^{k}\left(\frac{2^n}{2^{k+1}+2^n}\right)+\frac{1}{2}=\frac{k^2}{2}+\sum_{m=1}^{k}\left(\frac{2^{k+1}+2^m}{2^m+2^{k+1}}\right)+\frac{1}{2}=\frac{k^2}{2}+\sum_{m=1}^{k}(1)+\frac{1}{2}=\frac{k^2}{2}+k+\frac{1}{2}=\frac{k^2+2k+1}{2}=\frac{(k+1)^2}{2}$ So the statement is true for $k+1$, and hence true for all positive integers. Therefore,

$\sum_{m=1}^{2009}\left(\sum_{n=1}^{2009}\left(\frac{2^n}{2^m+2^n}\right)\right)=\frac{2009^2}{2}=2018040.5$

TB3) $(1+\sqrt{5})$

$S=\sqrt{8+\frac{8}{S}}$

$S^2=8+\frac{8}{S}$

$S^3=8S+8$

$S^3-8S-8=0$

By Descartes' Rule Of Signs, this polynomial has only one positive root.
It is easy to check that $S=1+\sqrt{5}$ works.