Solutions

1. For positive roots, we have: +++, so three sign changes: 3 or 1 positive roots.
   For negative roots, we have −++, so two sign changes: 2 or 0 negative roots.
   Choices are: 3+, 2-, 0i; 1+, 2-, 2i; 3+, 0-, 2i; 1+, 0-, 4i.
   The greatest number of possible imaginary roots is 4, E.

2. \(4x + 7 < 2x + 50 \rightarrow 2x < 43 \rightarrow x < 21.5\). Only D does not belong in the solution set.

3. \(x = \frac{8 \pm \sqrt{64 - 4(2)(1)}}{2(2)} = \frac{8 \pm 2\sqrt{14}}{4} = 2 \pm \sqrt{14}/2\). \(\sqrt{14}\) is between 3 and 4: A.

4. Sum of the reciprocal of the roots is \(-\) (linear coefficient)/constant term: \(-\frac{7}{-110} = \frac{7}{110}\), A.

5. Using expansion of minors, \(3(2x^2) - x(-x^3) + 6(-x^2 - 2) = 7 \rightarrow x^4 - 19 = 0\). The product of the roots is 19, E.

6. \(27^{x+3} = 81^{x+4} - 7 \rightarrow 3^{15x+9} = 3^{4x+28} \rightarrow 4|x+2| = 15x+37\). This gives \(x+2 = \frac{15}{4}x + \frac{37}{4}\) or \(x+2 = -\frac{15}{4}x - \frac{37}{4}\). The two solutions are \(-\frac{29}{11}\) and \(-\frac{45}{19}\), but \(-\frac{29}{11}\) is extraneous: D.

7. \(27^{x+2} = 9^{x+4} \rightarrow 3^{3x+8} = 3^{2x+14} \rightarrow 3|x+2|^2 = 2|x+2| + 4 \rightarrow 3|x+2|^2 - 2|x+2| - 4 = 0\).
   \(|x+2| = \frac{2 \pm \sqrt{4 - 4(3)(-4)}}{2(3)} = \frac{1 \pm \sqrt{13}}{3}\). Solving for \(x\) yields \(-5 \pm \frac{\sqrt{13}}{3}\) and \(-7 \pm \frac{\sqrt{13}}{3}\).
   The only viable solutions are \(-5 + \frac{\sqrt{13}}{3}\) and \(-7 - \frac{\sqrt{13}}{3}\), whose sum is \(-4\), B.

8. \(xy = x - y \rightarrow x(y-1) = -y \rightarrow x = \frac{y}{y-1}\). \(y\) cannot be 1, so correct answer is E.

9. The discriminant is 25, a perfect square, so the roots are rational, C.

10. \(\frac{5}{3+x} = x \rightarrow 5 = x^2 + 3x \rightarrow x = \frac{-3 \pm \sqrt{9 - 4(1)(-5)}}{2(1)} = \frac{-3 \pm \sqrt{29}}{3}\). Only the positive solution is feasible, A.

11. \(\frac{1}{|3x+1|} \geq 5 \rightarrow |3x+1| \leq \frac{1}{5} \rightarrow 3x+1 \leq \frac{1}{5}\) and \(3x+1 \geq -\frac{1}{5}\), \(x \neq -\frac{1}{3}\). This results in \(x \leq -\frac{4}{15} \cap x \geq -\frac{2}{5} : \left[-\frac{4}{15}, -\frac{1}{3}\right] \cup \left[-\frac{1}{3}, -\frac{4}{15}\right]\), E.
12. \[
\frac{1}{4} = \frac{1}{x} \rightarrow \frac{1}{4} = \frac{1}{x^2} \rightarrow x = \pm 2 .
\]
We consider only the positive solution, and C is the answer.

13. Since both sides must be positive, squaring each side will retain the positive values:
\[
4x^2 + 4x + 1 > x^2 - 10x + 25 \rightarrow 3x^2 + 14x - 24 > 0 \rightarrow (3x - 4)(x + 6) > 0 .
\]
This yields the intervals \((-\infty, -6) \cup \left(\frac{4}{3}, \infty\right)\). The closest integers that are also in these intervals are -7 and 2, whose sum is -5, A.

14. \[
4x^2 + 8x - 2\sqrt{4x^2 + 8x - 3} = 6 \rightarrow 4x^2 + 8x - 3 - 2\sqrt{4x^2 + 8x - 3} = 6 - 3 .
\]
Substituting \(y = 4x^2 + 8x - 3\), we have \(y - 2\sqrt{y - 3} = 0 \rightarrow (\sqrt{y - 3})(\sqrt{y + 1}) = 0 \rightarrow y = 9\). Now, \(4x^2 + 8x - 3 = 9 \rightarrow (x + 3)(x - 1) = 0 \rightarrow x = -3, x = 1\). E.

15. \[
x^2 - 5x < 6 \rightarrow x^2 - 5x - 6 < 0\] and \(x^2 - 5x + 6 > 0\). These two inequalities yield \((-1, 2)\) and \((-\infty, 2) \cup (3, \infty)\). The common solution is \((-1, 2) \cup (3, 6)\). The integers in the set are 0, 1, 4, and 5, whose sum is 10, A.

16. \[
\begin{aligned}
\log_x w &= 36 \\
\log_y w &= 18 \rightarrow x^{36} = w, \ y^{18} = w. \log_{xyz} w^2 &= 12 \rightarrow 2\log_{xyz} w = 12 \rightarrow \log_{xyz} w = 6.
\end{aligned}
\]
Now, \((xyz)^6 = w, \ x = w^{36}, \ y = w^{18}\). Now, \(\left(\frac{1}{w^{36}} \frac{1}{w^{18}} z\right)^6 = w \rightarrow w^2 z^6 = w \rightarrow z^6 = w^2 \rightarrow w = z^{12} .\)
\(\log_z w = 12\), B.

17. \[
\frac{\log(4x - 156)}{12x + 31} \leq 0\] tells us that \(x \neq -\frac{31}{12}\) and \(x > 39\) (by definition of logs).

For the numerator to be positive and the denominator negative, we have \(x > \frac{157}{4}, x < \frac{31}{12}\).

For the numerator to be negative and the denominator positive, we have \(x < \frac{157}{4}, x > -\frac{31}{12}\).

For the numerator to be 0, we have \(x = \frac{157}{4}\). There is no solution for the positive numerator, negative denominator case. For the log expression to be defined, \(x\) must be greater than 39. The final solution set is \(\left(39, \frac{157}{4}\right)\), the difference which is \(\frac{1}{4}\), D.

18. The point A, the point of tangency, and the center of the circle determine a right triangle with legs of length \(x\) and \(L\) and hypotenuse of length \(x + r\), where \(x\) is the shortest distance from
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A to the circle. Using the Pythagorean theorem, \(
\left(\frac{4}{3}r\right)^2 + r^2 = (x + r)^2 \rightarrow x = \frac{2}{3}r = \frac{L}{2}
\), C.

19. By assigning mass points, we can set up
the following system of equations:
\[
\begin{cases}
A + 1B = 2D & \rightarrow 4A + 4B = 8D \\
2C + 3B = 5E \\
5E + 8D = 13G
\end{cases}
\]
Combining the first two equations, we get
\((4A + 4B) + (2C + 3B) = (8D) + (5E)\) which
simplifies to \((4A + 2C) + 7B = 8D + 5E\).
Since we are using mass points, \(4A + 2C = 6F\).
By substitution, \(6F + 7B = 8D + 5E\). Again,
since we are using mass points, \(8D + 5E = 13G\).
Using substitution again, we have \(6F + 7B = 13G\),
so \(BG:GF\) is \(6:7\), E.

20. \(3^3 - 3^{3-3} = 78\sqrt{3} \rightarrow 3^3 \left(1 - 3^{-3}\right) = 78\sqrt{3} \rightarrow \frac{26}{27} \cdot 3^3 = 78\sqrt{3} \rightarrow 3^3 = 81\sqrt{3} = 3^4\cdot\frac{1}{3^2} = 3^2\), E.

21. \(\frac{7}{10} = 0.7\) and \(\frac{11}{15} \approx 0.733\), so no fraction with denominator of 1, 2, 3, 4, 5, or 6 will satisfy
this condition. \(\frac{5}{7}\) is the fraction with the smallest denominator that satisfies the inequality,
so D is the correct choice.

22. \(x\) and \(y\) must both be positive in order for \(x + y + xy = 54\). Trying \(x = 2\) gives a non-integer
value for \(y\). Trying \(x = 4\), we get \(y = 10\). By symmetry, we also get \((10, 4)\). No other
positive integer pairs satisfy the equation, so B, 14, is the sum of \(x\) and \(y\).

23. The graph of \(y = |x - 2| - 1\) has vertex \((2, -1)\) which moves to \((2, 1)\) on the graph of
\(y = \|x - 2| - 1\). Therefore, the line \(y = 1\), B, will intersect the graph three times: on the two
outermost oblique lines and at the vertex.

24. If the rabbit eats \(x\) kg of carrots per day, then \(x + 365x = 111 \rightarrow x = \frac{111}{366} = \frac{37}{122}\), C.

25. Let \(x\) be the free weight allowance and let \(y\) (in dollars) be the excess weight charge, per
kilogram, greater than \(x\). \(160 = \) (number of free kg)(cost of free kg) + (number of charged
kg)(cost of charged kg) = \((2x)(0) + (52 - 2x)y\). Alone, Hope would have had \(340 = (x)(0) +
(52 - x)y\). Combining these two equations, in terms of \(y\), we have:
\[y = \frac{160}{52 - 2x} = \frac{340}{52 - x}\]. Solving for \(x\), we get \(x = 18\), A.
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26. \( x^4 - 9x^2 + 4x + 12 < 0 \) factors into \((x + 3)(x + 1)(x - 2) < 0\), whose solution set is \((-3, \ -1)\). 
   \[-3 + (-1) = -4, \ \textbf{D}.\]

27. If \( x \) is the sufficient amount of bricks, then \( 0.93x = 10000 \rightarrow x = \frac{1000000}{93} \approx 10753 \). 
   10,800 bricks, \( \textbf{A} \), should be ordered.

28. Let the proportion of the circle the faster runner completes in 1 second be \( \frac{1}{x} \) and the proportion for the slower runner be \( \frac{1}{y} \). Then the times taken for a full lap are \( x \) and \( y \) seconds, respectively, so \( y - x = 10 \). In 12 minutes (720 seconds), the faster runner completes one extra lap, so we have \( \frac{720}{x} - \frac{720}{y} = 1 \rightarrow \frac{720}{x} - \frac{720}{x + 10} = 1 \). Solving, the only feasible solution for \( x \) is \( x = 80 \), \( \textbf{A} \).

29. Rewrite the given equations as \( a + b = -c \) and \((a + b)(a^2 - ab + b^2) + c^3 = 216 \). We also have \((a + b)^2 = (-c)^2 = a^2 + 2ab + b^2 = c^2 \). By substitution and after simplifying we have \((-c)(c^2 - 3ab) + c^3 = 216 \) which yields \( 3abc = 216 \rightarrow abc = 72 \), \( \textbf{B} \).

30. Begin with \( y = mx \) substitution.
   \[
   \begin{cases} 
   x^2 + xy + 3y^2 = 15 \\
   -5x^2 + 31xy - 3y^2 = 45 
   \end{cases} \rightarrow \begin{cases} 
   x^2 + mx^2 + 3m^2x^2 = 15 \\
   -5x^2 + 31mx^2 - 3m^2x^2 = 45 
   \end{cases} \rightarrow \begin{cases} 
   1 + m + 3m^2 = 15 \\
   -5 + 31m - 3m^2 = 45 
   \end{cases}
   
   After multiplying the top equation by 3 and equating the expressions, we have 
   \( -5 + 31m - 3m^2 = 3 + 3m + 9m^2 \rightarrow (3m-1)(m-2) = 0 \), so \( y = \frac{1}{3}x \) or \( y = 2x \).

   Substituting with \( y = \frac{1}{3}x \), we get \( x^2 + \frac{1}{3}x^2 + \frac{1}{3}x^2 = 15 \rightarrow x = \pm 3 \Rightarrow (3, 1), \ (-3, -1) \).

   Substituting with \( y = 2x \), we get \( x^2 + 2x^2 + 12x^2 = 15 \rightarrow x = \pm 1 \Rightarrow (1, 2), \ (-1, -2) \).

   The Quadrant I solutions are (3, 1) and (1, 2). The distance between them is \( \sqrt{5} \), \( \textbf{C} \).

TB1. \( 6241 = r^2 \rightarrow r = 79 \). Therefore, the circumference is \( 158\pi \).
TB2. The lines will intersect at \((0, 2), \left(\frac{4}{3}, -2\right), \left(-\frac{4}{3}, -2\right)\). Two of the legs have length \(\frac{4}{3} \sqrt{10}\) and the third leg has length \(\frac{8}{3}\). The most descriptive name is isosceles triangle or acute isosceles triangle.

TB3. \(1 = 10^6 x^{\frac{-3}{2}} = x^{\frac{1}{2}} = 10^2 \rightarrow x = 10000\) .