Solutions:

1. \( B^T = \begin{bmatrix} 1 & 0 & -3 \\ -2 & 3 & 1 \end{bmatrix} \), \( AB^T = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ -2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \end{bmatrix} \). C

2. \( \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ a & b \end{bmatrix} = \begin{bmatrix} 1 + a^2 & ab \\ ab & b^2 \end{bmatrix} \). Since the resulting matrix is the identity matrix, \( ab = 0 \) and \( 1 + a^2 = b^2 = 1 \). a = 0 and \( b = \pm 1 \). Take the absolute value and get (0, 1). D

3. \( \begin{bmatrix} 1 & 2 & -3 & 0 \\ -2 & 1 & 0 & 3 \\ 0 & 4 & -4 & -1 \\ 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -3 & 0 \\ -2 & 1 & 0 & 3 \\ 0 & 4 & -4 & -1 \\ 0 & -1 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 28 \\ 0 & 4 & 19 \\ -1 & 2 & 5 \end{bmatrix} = -1(-36) = 36 \). C

4. \( x \) has roots 2 and -4, y is -7. Using the positive root of \( x \), \( xy = (-7)(2) = -14 \). E

5. Evaluate \( \frac{1}{2} \begin{bmatrix} 2 & -3 & 1 \\ 3 & -5 & 1 \\ 1 & 4 & 1 \end{bmatrix} \rightarrow \frac{1}{2} \left| (-10 - 3 + 12) - (-5 + 8 - 9) \right| \rightarrow \frac{5}{2} \). B

6. \( \begin{vmatrix} x & 0 & c \\ -1 & x & b \\ 0 & -1 & a \end{vmatrix} = 0 \rightarrow ax^2 + bx + c \). Sum of roots of \( x \) is \( \frac{-b}{a} \), and product of roots of \( x \) is \( \frac{c}{a} \). C

7. \( \begin{vmatrix} -5 & 9 \\ 7 & 6 \end{vmatrix} = (-30 - 63) = -93 \). B

8. \( \text{Det } \begin{vmatrix} 4 & 5 & -1 \\ -3 & 2 & 0 \\ 3 & -2 & 1 \end{vmatrix} = 4(2) - 5(-3) - 1(6 - 6) = 23 \rightarrow b = 23 \). \( A^{-1} = \frac{1}{23} \begin{bmatrix} 2 & -3 & 2 \\ 3 & 7 & 3 \\ 0 & 23 & 23 \end{bmatrix} \). E

9. Using elementary row operations

\[
\begin{align*}
&\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \\ 0 & -1 & 5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \\
&\begin{bmatrix} 1 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \\ 0 & -1 & 5/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{bmatrix} \\
\end{align*}
\]
10. \[
\begin{bmatrix}
-2 & -3 \\
1 & 2
\end{bmatrix}^{-1} = -1 \begin{bmatrix}
2 & 3 \\
-1 & -2
\end{bmatrix}; \quad \begin{bmatrix}
-2 & -3 \\
1 & 2
\end{bmatrix} \begin{bmatrix}
5 & 3 \\
-2 & 1
\end{bmatrix} = \begin{bmatrix}
-4 & -9 \\
1 & 5
\end{bmatrix}
\]

11. Singular matrices have determinants of 0. \[
\begin{bmatrix}
1 & 4 & 2 \\
3 & 3 & 2 \\
-1 & y & 0
\end{bmatrix} = 0 \rightarrow (-8 + 6y) - (-6 + 2y) = 0, \ y = \frac{1}{2}
\]

12. \[
\begin{bmatrix}
\frac{1}{2} & -\frac{\sqrt{3}}{2} \\
\frac{\sqrt{3}}{2} & \frac{1}{2}
\end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}. \quad 60^\circ \text{ counterclockwise.}
\]

13. \[
det \begin{bmatrix}
i & 4 & -2 \\
-3 & i & 4
\end{bmatrix} = 2i(16 + 2i) + i(4i - 6) + 3(-1 + 12) = 32i - 4 - 6i - 3 + 36 = 25 + 26i.
\]

14. To solve for \( z \), the coefficients of \( z \) in the third column are replaced.

15. Singular matrices are matrices with determinant of 0. I, II, V.

16. \[
\begin{bmatrix}
1-x & 3 \\
3 & 1-x
\end{bmatrix} \rightarrow (1-x)(1-x) - 9 = 0 \rightarrow x = 4, -2.
\]

17. The resulting matrix is a 3x2 matrix; 6 elements.

18. \[
\begin{bmatrix}
3 & -1 & 2 \\
1 & 2 & -K
\end{bmatrix} = 0 \rightarrow (K + 12 - 2) - (-2 + 4 + 3K) = 0 \rightarrow K = 4
\]

19. \[
\frac{4 \cdot 2 + 3 \cdot 1 - 2 \cdot 1 + 3}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{12}{\sqrt{9}} = 4.
\]

20. \[
\begin{bmatrix}
1 & 4 \\
2 & 3 \\
3 & 2
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 2 \\
2 & 3 & 1
\end{bmatrix}^T = \begin{bmatrix}
9 & 12 & 6 \\
8 & 9 & 7 \\
7 & 6 & 8
\end{bmatrix}^T = \begin{bmatrix}
9 & 8 & 7 \\
12 & 9 & 6 \\
6 & 7 & 8
\end{bmatrix}.
\]

21. A
22. C

23. D

\[
\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}^{11} \rightarrow \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}\left(\begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}\right)^{5} \rightarrow \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}\begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix}^{2} \rightarrow \begin{bmatrix}
1 & -1 \\
1 & 1
\end{bmatrix}\begin{bmatrix}
0 & -2 \\
2 & 0
\end{bmatrix}\begin{bmatrix}
-4 \\
0
\end{bmatrix}^{2}
\]
\[
\begin{pmatrix}
1 & -1 \\
1 & 1
\end{pmatrix}\begin{pmatrix}
0 & -2 \\
2 & 0
\end{pmatrix}\begin{pmatrix}
-4 \\
0
\end{pmatrix}^{2} \rightarrow \begin{pmatrix}
-32 & -32 \\
32 & -32
\end{pmatrix}.
\]

25. \(3 \cdot 3^3 = 81\) C

26.

\[
\begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
1 & 0 & 6
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
1 & 0 & 6
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
1 & 0 & 6
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
1 & 0 & 6
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 2 & 3 \\
0 & 4 & 5 \\
1 & 0 & 6
\end{pmatrix}\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

27. A 2x2 and a 3x2 matrix cannot be multiplied in that order. E

28. \(3a + 6 = -6\) \(-18\), \(a = -10\). B

29. \([(2)(1)(1) + (-1)(-2)(1) + (-3)(2)(0)] - [(0)(1)(1) + (2)(2)(-2) + (-3)(-1)(1)] = 9\). C

30. \[
\begin{pmatrix}
i & \pi \\
e & 1
\end{pmatrix}^{-1} = \frac{1}{i-e\pi}\begin{pmatrix}1 & -\pi \\
e\pi-i & e
\end{pmatrix} = \frac{1}{e\pi-i}\begin{pmatrix}1 & -\pi \\
-e & i
\end{pmatrix}.
\]

Tiebreakers:

1. \(\begin{bmatrix}1 & 0 & 2 \\
0 & 0 & 3 \\
2 & 1 & 0\end{bmatrix}\); The sum of the entries gives the directed paths, so 9 directed paths

2. \(\begin{bmatrix}3 & -2 & 5 \end{bmatrix} \cdot \begin{pmatrix}4 \\
6 \\
10
\end{pmatrix} = \begin{pmatrix}50
\end{pmatrix}\)
2009 Matrices and Determinants Theta Topic Test

3. \[
\begin{bmatrix}
-2 & 1 & 0 \\
5 & -3 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
a & 0 \\
3 & b
\end{bmatrix}
= \begin{bmatrix}
-3 & 10 \\
8 & -31
\end{bmatrix}
= \begin{bmatrix}
-2a+3 & b \\
5a-7 & -3b-1
\end{bmatrix}
= \begin{bmatrix}
-3 & 10 \\
8 & -31
\end{bmatrix}; a = 3, b = 10.
\]

\[a + b = 13.\]