Mu Limits and Derivatives Solutions Nationals 2010

1. Let $u = 3x - 2$. This makes the limit
$\lim_{u\to\infty} u\ln\left(\frac{u+1}{u}\right) = \lim_{u\to\infty} \ln\left(1+\frac{1}{u}\right)^u = \ln\left(\lim_{u\to\infty}\left(1+\frac{1}{u}\right)^u\right) = \ln e = 1$. B

2. $f(x) = \sin x,\ x = \frac{\pi}{3},\ h = \frac{3600'}{\pi} = \frac{60^\circ}{\pi} = \frac{1}{3}$ radians and $f'(x) = \cos x$.
Thus we have $\sin\left(60^\circ\left(3600/\pi\right)'\right) = \sin\left(\frac{\pi}{3}\right) + \frac{1}{3}\cos\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} + \frac{1}{6} = \frac{3\sqrt{3}+1}{6}$. B

3. The error in the approximation is equal to
$\frac{d\left(\sqrt{x}\right)}{dx} = \frac{1}{2\sqrt{x}} \le .001 \to \sqrt{x} \ge 500 \to x \ge 250000$. D

4. The terms with $\ln x$ do not affect the limit thus it is equivalent to $\lim_{x\to\infty}\frac{6x}{x} = 6$.
C

5. $\lim_{n\to\infty}\frac{F_{n+2}}{F_n} = \lim_{n\to\infty}\frac{F_{n+1}+F_n}{F_n} = \lim_{n\to\infty}\frac{F_{n+1}}{F_n} + 1 = \frac{1+\sqrt{5}}{2} + 1 = \frac{3+\sqrt{5}}{2}$. C

6. $f(x) = \frac{1}{1-\cos x}$ and $f'(x) = \frac{-\sin x}{(1-\cos x)^2}$. Thus $f'\left(\frac{\pi}{3}\right) = \frac{-\sqrt{3}/2}{(1-1/2)^2} = -2\sqrt{3}$. A

7. I) Indeterminate
II)This reduces to ∞^0 which is indeterminate
III) Indeterminate
IV) Indeterminate
D

8. The tangent line is given by $y - e^{\pi/2} = e^{\pi/2}\left(x - \frac{\pi}{2}\right)$ which has the root
$x_1 = \frac{\pi}{2} - 1$. The normal line is given by $y - e^{\pi/2} = -e^{-\pi/2}\left(x - \frac{\pi}{2}\right)$ which has the root $x_2 = \frac{\pi}{2} + e^{\pi}$.
Thus $C = x_2 - x_1 = e^{\pi} + 1$. Using the distance formula using the roots of the lines and the given point $\left(\frac{\pi}{2}, e^{\pi/2}\right)$ yields
$A = \sqrt{\left(e^{\pi/2}\right)^2 + \left(\frac{\pi}{2} - \frac{\pi}{2} + 1\right)^2} = \sqrt{e^{\pi} + 1}$ and
$B = \sqrt{\left(-e^{\pi/2}\right)^2 + \left(\frac{\pi}{2} + e^{\pi} - \frac{\pi}{2}\right)^2} = \sqrt{e^{\pi} + e^{2\pi}}$.

Finally $\frac{A^2+B^2-C}{e^\pi}=\frac{e^\pi+1+e^\pi+e^{2\pi}-e^\pi-1}{e^\pi}=e^\pi+1$. B

9. Function I has a domain of $[4,\infty)$ and the limit exists only from the right side and is thus not completely differentiable. Function II is continuous and differentiable for all values. Function III has a cusp and is thus not differentiable. The absolute value in Function IV is moot because cosine is an even function. It is differentiable for all values. Thus the answer is 2. B

10. $\frac{dy}{dx}=\frac{-F_x}{F_y}\rightarrow\frac{dx}{dy}=\frac{-F_y}{F_x}$. $F_x=3x^2y^2+2xy\cos(y^2)$ and $F_y=2yx^3-2x^2y^2\sin(y^2)+2x\cos(y^2)$. Substituting for y in the original equation yields $x^3\pi+x^2\sqrt{\pi}\cos\pi=0\rightarrow x^2(x\pi-\sqrt{\pi})\rightarrow x=\frac{\sqrt{\pi}}{\pi}$
$F_x\left(\frac{\sqrt{\pi}}{\pi},\sqrt{\pi}\right)=3+2(1)(-1)=1$ and $F_y\left(\frac{\sqrt{\pi}}{\pi},\sqrt{\pi}\right)=\frac{2}{\pi}-0-\frac{2\sqrt{\pi}}{\pi}=\frac{2-2\sqrt{\pi}}{\pi}$.
$\frac{dx}{dy}=\frac{-F_y}{F_x}=\frac{2\sqrt{\pi}-2}{\pi}$. A

11. The plane and tracking station make the hypotenuse of a right triangle with length 20. The height is 12 and we can solve for the horizontal distance between the plane and station as $\sqrt{20^2-12^2}=16$. Using the Pythagorean theorem we have a relation between the sides such that $S^2=x^2+y^2$ where x is the horizontal distance and y is the vertical distance. Differentiating yields $S\frac{dS}{dt}=x\frac{dx}{dt}+y\frac{dy}{dt}$. $\frac{dS}{dt}=400$ and $\frac{dy}{dt}=0$ since the plane does not move up or down vertically. Thus solving for $\frac{dx}{dt}$ gives $\frac{dx}{dt}=500$. C

12. $\frac{dx}{dt}=\cos t$ and $\frac{dy}{dt}=-\sin t$ thus we have $\frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{-\sin t}{\cos t}=-\tan t$. The chain rule must be used to find the second derivative so that we have $\frac{d^2y}{dx^2}=\frac{d\left(\frac{dy}{dx}\right)}{dt}\frac{dt}{dx}$. This means $\frac{d^2y}{dx^2}=-\sec^2 t\bullet\frac{1}{\cos t}=-\sec^3 t$. Evaluated at $t=\pi$ we have $-\sec^3\pi=-(-1)^3=1$. B

13. The law of the mean is $\dfrac{f(b)-f(a)}{b-a}=f'(x_0)$.

$\dfrac{f(4)-f(1)}{3}=\dfrac{47-2}{3}=15=f'(x_0)=6x_0\to x_0=\dfrac{5}{2}$. B

14. $f'(x)=-\sin x+\cos x\to f(0)=1$ and $f(0)=6$. The approximation is $y=x+6\to\cos(-.5)+\sin(-.5)+5\approx-.5+6=11/2$. E

15. In one period of $k(x)=\cos(2010x)+\sin(1005x)$ both $\cos(2010x)$ and $\sin(1005x)$ must go through an integral number of periods. The period of $\cos(2010x)$ is $\dfrac{\pi}{1005}$ and the period of $\sin(1005x)$ is $\dfrac{2\pi}{1005}$. Thus the period of $k(x)$ is $T=\dfrac{m\pi}{1005}=\dfrac{2n\pi}{1005}$ for some values of m and n. The smallest solutions are $(m,n)=(2,1)$ and the period of $k(x)$ is $\dfrac{2\pi}{1005}$. For the equation of the line we have $A=2\pi$, $B=1005$, and $C=\dfrac{2\pi}{1005}$ and $y=2\pi x^2+1005x+\dfrac{2\pi}{1005}$.

Differentiating yields $y'=4\pi x+1005\to x=\dfrac{-1005}{4\pi}$. D

16. In general we have $\lim\limits_{h\to0}\dfrac{f(x+ah)-f(x+bh)}{h}=(a-b)f'(x)$. Breaking up the given limit reveals a similar situation:

$\lim\limits_{h\to0}\dfrac{f(x-3h)-f(x+3h)}{h}+\lim\limits_{h\to0}\dfrac{f(x+2h)-f(x-4h)}{h}$.

$\lim\limits_{h\to0}\dfrac{f(x-3h)-f(x+3h)}{h}=-6f'(x)$ and $\lim\limits_{h\to0}\dfrac{f(x+2h)-f(x-4h)}{h}=6f'(x)$.

The sum of the limits is 0. D

17. $\lim\limits_{x\to0^\circ}\dfrac{\cot x}{\cot 2x}=\lim\limits_{x\to0^\circ}\dfrac{\tan 2x}{\tan x}=\lim\limits_{x\to0}\dfrac{\tan\left(\frac{\pi x}{90}\right)}{\tan\left(\frac{\pi x}{180}\right)}=\lim\limits_{x\to0}\dfrac{\frac{\pi}{90}\sec^2\left(\frac{\pi x}{90}\right)}{\frac{\pi}{180}\sec^2\left(\frac{\pi x}{180}\right)}=2$. C

18. $f'(x)=xe^x+e^x-2010e^x$

$f''(x)=xe^x+2e^x-2010e^x$

$f^n(x)=xe^x+ne^x-2010e^x$

$f^{2010}(x)=xe^x+2010e^x-2010e^x=xe^x$

A

19. $a_n=\sqrt{20-a_{n-1}}$ and as $n\to\infty$, $a_n=a_{n-1}$ thus we have $a_n=\sqrt{20-a_n}\to a_n^2+a_n-20=0$. Solving for a_n gives $a_n=\dfrac{-1\pm\sqrt{81}}{2}=4,-5$. Since the square root must be positive the answer is 4. B

20. $y=\sqrt{x-y} \to y^2+y=x$ and by completing the square we have $y=\sqrt{x+1/4}-1/2$

$g'(1)=\dfrac{1}{f'(g(1))}$. To solve for $g(1)$ we have $\sqrt{x+1/4}-1/2=1 \to x=2$. So

$g'(1)=\dfrac{1}{f'(2)}$ and $f'(x)=\dfrac{1}{2}\left(x+\dfrac{1}{4}\right)^{-1/2}$ thus $g'(1)=\dfrac{1}{\dfrac{1}{2}\left(\dfrac{5}{4}\right)^{-1/2}}=\sqrt{5}$. C

21. Critical values occur where $f'(x)=0$ or is undefined at values of x in the domain of $f(x)$.

$f'(x)=\dfrac{(x-3)(2x-5)-(x^2-5x+1)}{(x-3)^2}=\dfrac{x^2-6x+14}{(x-3)^2}$. This has no real roots and is undefined at a point not within the domain of $f(x)$. Thus A=0.

The product AB is thus automatically 0. A

22. Newtons Method : $x_{n+1}=x_n-\dfrac{f(x_n)}{f'(x_n)}$. $f(x)=x^3+4x-5$ and $f'(x)=3x^2+4$.

$$x_1=0, x_2=x_1-\frac{f(x_1)}{f'(x_1)}=0-\frac{-5}{4}=\frac{5}{4}, x_3=\frac{5}{4}-\frac{\frac{125}{64}}{\frac{139}{16}}=\frac{285}{278}$$. D

23. $\lim\limits_{x\to\infty} x^{(1/x)}=y \to \ln y=\lim\limits_{x\to\infty}\dfrac{1}{x}\ln x=\lim\limits_{x\to\infty}\dfrac{1}{x}=0 \to y=e^0=1$. C

24. $y=x^{1/3} \to y'=\dfrac{1}{3}x^{-2/3}$ and since h = -1 we have

$\sqrt[3]{63}\approx 4-1\left(\dfrac{64^{-2/3}}{3}\right)=4-\dfrac{1}{48}=\dfrac{191}{48}$. B

25. $y=x^2-x^y$ and $\dfrac{dy}{dx}=2x-x^y\ln x$. Evaluated at x = 1, y = 0 we have $\dfrac{dy}{dx}=2$. C

26. $\lim\limits_{x\to0}\dfrac{\sin x}{|x|}=\begin{cases}\lim\limits_{x\to0^+}\dfrac{\sin x}{x}=1\\ \lim\limits_{x\to0^-}\dfrac{-\sin x}{x}=-1\end{cases}$. Thus the limit does not exist. D

27. $\lim_{k\to 0^+} \frac{\sqrt{k}}{\sqrt{16+\sqrt{k}}-4} = \lim_{k\to 0^+} \frac{\sqrt{k}}{\sqrt{16+\sqrt{k}}-4} \cdot \frac{\sqrt{16+\sqrt{k}}+4}{\sqrt{16+\sqrt{k}}+4} = \lim_{x\to 0^+} \sqrt{16+\sqrt{k}}+4 = 8$. C

28.
$f(x) = \frac{\cos x}{x} - x \to f'(x) = \frac{-x\sin x - \cos x}{x^2} - 1 \to f''(x) = \frac{x^2(-x\cos x - \sin x + \sin x) - 2x(-x\sin x - \cos.}{x^4}$
$= \frac{(2-x^2)\cos x + 2x\sin x}{x^3}$. A

29. When x = 1 the two equations are equal thus we have
$1 - 2(k+1) + 2 = 0 \to k = \frac{1}{2}$.
$Q'(x) = \begin{cases} 2x - 2(k+1), & x \ge 1 \\ 1 & , x < 1 \end{cases}$. Equating these equations when x = 1 gives
$2 - 2(k+1) = 1 \to k = -\frac{1}{2}$.
Since two different values are obtained there is no such value of k. D

30. $\frac{dy}{dx}\left(\int_w^u f(x)\,dx\right) = u'f(u) - w'f(w)$. Thus
$\frac{dy}{dx}\left(\int_{x^2}^{\sin x} \sqrt{1-x^2}\,dx\right) = \cos x\sqrt{1-\sin^2 x} - 2x\sqrt{1-x^4} = \cos^2 x - 2x\sqrt{1-x^4}$. A