1. $\begin{bmatrix} 7 & -2 \\ 9 & 1 \end{bmatrix} + \begin{bmatrix} -3 & 8 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 7-3 & -2+8 \\ 9+2 & 1+0 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 11 & 1 \end{bmatrix}$. **A**

2. $\langle 1,-1,1 \rangle \cdot \langle 2,5,-2 \rangle = 1\cdot 2 + -1\cdot 5 + 1\cdot -2 = 2-5-2 = -5$. **A**

3. If every entry of a row of a matrix **A** is multiplied by a scalar *c*, then the determinant of the new matrix is the determinant of **A** times *c*. If every entry of a 4×4 matrix is multiplied by 2, then each of it's four rows are multiplied by 2. Therefore $|\mathbf{B}| = 2^4 |\mathbf{A}|$ and hence $\frac{|\mathbf{B}|}{|\mathbf{A}|} = 16$, noting that $|\mathbf{A}| \neq 0$ since **A** is nonsingular. **D**

4. The entry in the *i*th row and *j*th column of $\mathbf{E}_{23} \cdot \mathbf{E}_{35}$ is $\sum_{k=1}^{n} a_{ik} b_{kj}$, where the entry in the *i*th row and *j*th column of $\mathbf{E}_{23}$ and $\mathbf{E}_{35}$ are a_{ij} and b_{ij}, respectively. The only one of the sums that will have a non-zero entry will occur when $i=2$, $j=5$, and $k=3$. Therefore the entry in the 2nd row and 5th column of $\mathbf{E}_{23} \cdot \mathbf{E}_{35}$ is 1, and all other entries are 0. Therefore $\mathbf{E}_{23} \cdot \mathbf{E}_{35} = \mathbf{E}_{25}$. **A**

5. Expanding by minors, $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -2+1 = -1$. **B**

6. The two curves intersect in the *x*-*y* plane when $x^2+3 = 2x+6$. $x^2-2x-3=0$. $x=-1$ and $x=3$. The area of *R* is $\int_{-1}^{3} \left(2x+6-x^2-3\right) dx = \int_{-1}^{3} \left(3+2x-x^2\right) dx = \left[3x+x^2-\frac{x^3}{3}\right]_{-1}^{3} = 9+\frac{5}{3} = \frac{32}{3}$. When *R* is transformed to R', the area of R' will be the area of *R* multiplied by $\begin{vmatrix} 4 & 2 \\ 1 & 2 \end{vmatrix} = 6$. The area of R' is therefore 64. **D**

7. In order for two vectors to be linearly independent, they must not be scalar multiples of each other. Each pair of vectors are scalar multiples except for $\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ 10 \\ 3 \end{pmatrix}$. **B**

8. Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$, noting that **A** is symmetric. $\mathbf{x}^T \mathbf{A} \mathbf{x} = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 & a_{12}x_1 + a_{22}x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 7x_1^2 - 4x_1x_2 + 5x_2^2$. Equating coefficients, we have $2a_{12} = -4$ and hence $a_{12} = -2$. **B**

9. Since the coefficients of $f(x)$ are real numbers and exactly one root is real, the other two must be complex conjugates. Let the roots be $r = a+bi$, $s = a-bi$, and t where a, b, and t are real numbers. Then $\delta = \begin{vmatrix} 1 & a+bi & (a+bi)^2 \\ 1 & a-bi & (a-bi)^2 \\ 1 & t & t^2 \end{vmatrix} = \begin{vmatrix} 1 & a+bi & a^2-b^2+2abi \\ 1 & a-bi & a^2-b^2-2abi \\ 1 & t & t^2 \end{vmatrix}$. First notice that $\delta = 0$ if and only if $b = 0$ and $a = t$, which is not the case. Also, notice that subtracting the second row from the first row will leave the determinant unchanged, so

$$\delta = \begin{vmatrix} 0 & 2bi & 4abi \\ 1 & a-bi & a^2-b^2-2abi \\ 1 & t & t^2 \end{vmatrix} = -2bt^2i + 4abti + 2a^2bi - 2b^3i + 4ab^2 - 4a^2bi - 4ab^2$$

$= -2bt^2i + 4abti - 2a^2bi - 2b^3i$. This is a non-zero, purely imaginary number, so its square will be a negative real number. **B**

10. Notice that for any invertible matrix **A**, $|\mathbf{A}||\mathbf{A}^{-1}| = 1$. However, since all of the entries of both **A** and $\mathbf{A}^{-1}$ are integers, $|\mathbf{A}|$ and $|\mathbf{A}^{-1}|$ must also be integers. If the product of two integers is 1, then either both integers are 1 or both integers are -1. **B**

11. $\|\mathbf{v}\| = \sqrt{\left(\frac{1}{3}\right)^2 + a^2} = 1$. $\frac{1}{9} + a^2 = 1$. $a^2 = \frac{8}{9}$. Since a is positive, we have $a = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$. **C**

12. The trace of a matrix is the sum of the diagonal elements. The trace is therefore 9. **D**

13. Putting **A** is reduced row echelon form gives $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. The rank is the number of non-zero rows, which is 1. **B**

14. We want to find the vectors $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Then we must have $x + 2y - z = 0$. $x = -2y + z$. We can take y and z as free variables. Setting $y = 1$ and $z = 0$, we get $x = -2$. Setting $y = 0$ and $z = 1$, we get $x = 1$. Then two linearly independent solutions are $\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. The only choice that is not on the span of these two vectors is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. This can be checked by noticing that $\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ -2 & -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. **D**

15. $\mathbf{B}^T$ is a 2×3 matrix, so $\mathbf{B}^T\mathbf{A}$ is defined, and the product is a 2×4 matrix. **B**

16. Notice that $\mathbf{B} = \mathbf{A}\mathbf{A}^T$. Therefore $|\mathbf{B}| = |\mathbf{A}\mathbf{A}^T| = |\mathbf{A}||\mathbf{A}^T| = |\mathbf{A}|^2 = 4^2 = 16$. **D**

17. Statement I is true. If **A** is invertible, then the system $\mathbf{Ax}=\mathbf{0}_{n\times 1}$ has only the trivial solution, and hence statement II is true. Since $0=|\mathbf{A}|=|\mathbf{A}^T|$, statement III is also true. Statement IV is false. Consider the matrix $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ which is singular, but has trace 1. Three of the statements are true. **C**

18. Notice that $\cos(\theta)=\dfrac{\langle 1,1\rangle\cdot\langle 1+t,1-t\rangle}{\|\langle 1,1\rangle\|\|\langle 1+t,1-t\rangle\|}=\dfrac{2}{\sqrt{2}\sqrt{2t^2+2}}=\dfrac{1}{\sqrt{t^2+1}}$. Differentiating both sides with respect to *t*, $-\sin(\theta)\dfrac{d\theta}{dt}=-\dfrac{1}{2}\left(t^2+1\right)^{-\frac{3}{2}}\cdot 2t=-t\left(t^2+1\right)^{-\frac{3}{2}}$. Notice that when $t=1$, $\cos(\theta)=\dfrac{\sqrt{2}}{2}$ and hence $\sin(\theta)=\dfrac{\sqrt{2}}{2}$ since θ is acute. Then when $t=1$ $\dfrac{d\theta}{dt}=-\sqrt{2}\cdot-(2)^{-\frac{3}{2}}=\dfrac{1}{2}$. **C**

19. $D(x)=\begin{vmatrix}x & 1\\ e^x & x+1\end{vmatrix}=x(x+1)-e^x=x^2+x-e^x$. $D'(x)=2x+1-e^x$. $D'(0)=0+1-1=0$. **E**

20. Let $\mathbf{x}=\begin{bmatrix}x\\ y\end{bmatrix}$ and $x^2+y^2=1$. Then $\mathbf{Ax}=\begin{bmatrix}x\\ 2y\end{bmatrix}$. We would like to maximize $\sqrt{x^2+4y^2}$. $x^2=1-y^2$, so we can maximize $\sqrt{1+3y^2}$. Realizing that $-1\le y\le 1$, this is clearly maximized when either $y=1$ or $y=-1$, and the maximum value is $\sqrt{4}=2$. **D**

21. $\mathbf{A}^3=\mathbf{A}\cdot\mathbf{A}^2=\mathbf{A}\left(\mathbf{A}+2\mathbf{I}_n\right)=\mathbf{A}^2+2\mathbf{A}=\mathbf{A}+2\mathbf{I}_n+2\mathbf{A}=3\mathbf{A}+2\mathbf{I}_n$. **C**

22. $\|\mathbf{v}\times\mathbf{i}\|=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & x & y\\ 1 & 0 & 0\end{vmatrix}=y\mathbf{j}-x\mathbf{k}=1$. Therefore $x^2+y^2=1$. $\|\mathbf{v}\|=\sqrt{1^2+x^2+y^2}=\sqrt{2}$. **B**

23. Row reducing the augmented matrix for the system of equations, $\begin{bmatrix}2 & -6 & 8\\ -3 & 9 & -12\end{bmatrix}\to\begin{bmatrix}1 & -3 & 4\\ -1 & 3 & -4\end{bmatrix}\to\begin{bmatrix}1 & -3 & 4\\ 0 & 0 & 0\end{bmatrix}$. Since the second row is all zeros, the system of equations can be represented as a single equation in two variables (this is because the second equation is a scalar multiple of the first). Since there are more variables than there are linearly independent equations, there are infinitely many solutions. **D**

24. $\|\langle t,2+t\rangle\|=\sqrt{t^2+(2+t)^2}=\sqrt{2t^2+4t+4}$. For simplicity, we can find the *t* that will minimize $2t^2+4t+4$, realizing that this value of *t* will also minimize the square root. Taking the derivative and setting it equal to 0, we get $4t+4=0$, and hence $t=-1$. Notice that the second derivative is $4>0$, and hence this does indeed lead to a minimum. **E**

25. By plotting the vectors in the *x*-*y* plane and using the right-hand rule, it can be seen that only the set of vectors $\mathbf{u}=\langle 5,1,0\rangle$ and $\mathbf{v}=\langle 2,3,0\rangle$ will satisfy $\mathbf{u}\times\mathbf{v}=\langle 0,0,a\rangle$, where $a>0$. **A**

26. Let *x* be the price of an apple and *y* be the price of an orange. We then want to solve for *y* where $4x+7y=43$ and $3x+y=11$. Using Cramer's Rule, $x=\dfrac{\begin{vmatrix}4 & 43\\ 3 & 11\end{vmatrix}}{\begin{vmatrix}4 & 7\\ 3 & 1\end{vmatrix}}=\dfrac{44-132}{4-21}=\dfrac{-88}{-17}=5$. **D**

27. $\begin{bmatrix} -1 & 2 \\ 8 & -1 \end{bmatrix}\mathbf{x} = \lambda\mathbf{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\mathbf{x}$. Then $\begin{bmatrix} -1-\lambda & 2 \\ 8 & -1-\lambda \end{bmatrix}\mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Such a system of equations can only have a non-zero solution if the matrix $\begin{bmatrix} -1-\lambda & 2 \\ 8 & -1-\lambda \end{bmatrix}$ is singular, that is, $\begin{vmatrix} -1-\lambda & 2 \\ 8 & -1-\lambda \end{vmatrix} = 0$. Then $\lambda^2 + 2\lambda - 15 = 0$ and $\lambda = 3$ or $\lambda = -5$. **C**

28. The cofactor is $(-1)^{2+3}\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = -3$. **B**

29. $|\mathbf{A}| = |\mathbf{A}^T| = |-\mathbf{A}| = (-1)^{11}|\mathbf{A}| = -|\mathbf{A}|$. As a result, we must have that $|\mathbf{A}| = 0$. **A**

30. $\|\langle 1,2,3\rangle - 2\cdot\langle -1,4,3\rangle\| = \|\langle 3,-6,-3\rangle\| = \sqrt{3^2 + (-6)^2 + 3^2} = \sqrt{9+36+9} = \sqrt{54} = 3\sqrt{6}$. **C**