1. \[
\begin{bmatrix}
7 & -2 \\
9 & 1
\end{bmatrix} +
\begin{bmatrix}
-3 & 8 \\
2 & 0
\end{bmatrix} = 
\begin{bmatrix}
7-3 & -2+8 \\
9+2 & 1+0
\end{bmatrix} = 
\begin{bmatrix}
4 & 6 \\
11 & 1
\end{bmatrix}.
\]

2. \[
\langle 1, -1, 1 \rangle \cdot \langle 2, 5, -2 \rangle = 1 \cdot 2 + (-1) \cdot 5 + 1 \cdot (-2) = 2 - 5 - 2 = -5.
\]

3. If every entry of a row of a matrix \(A\) is multiplied by a scalar \(c\), then the determinant of the new matrix is the determinant of \(A\) times \(c\). If every entry of a \(4 \times 4\) matrix is multiplied by 2, then each of its four rows are multiplied by 2. Therefore \(|B| = 2^4|A|\) and hence \[^{B}A|=16\], noting that \(|A| \neq 0\) since \(A\) is nonsingular.

4. The entry in the \(i\)th row and \(j\)th column of \(E_{23} \cdot E_{35}\) is \(\sum_{k=1}^{n} a_{ik} b_{kj}\), where the entry in the \(i\)th row and \(j\)th column of \(E_{23}\) and \(E_{35}\) are \(a_{ij}\) and \(b_{ij}\), respectively. The only one of the sums that will have a non-zero entry will occur when \(i = 2\), \(j = 5\), and \(k = 3\). Therefore the entry in the 2\(^{nd}\) row and 5\(^{th}\) column of \(E_{23} \cdot E_{35}\) is 1, and all other entries are 0. Therefore \(E_{23} \cdot E_{35} = E_{25}\).

5. Expanding by minors,
\[
\begin{vmatrix}
1 & 1 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{vmatrix} = \begin{vmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 0 & 1
\end{vmatrix} = -2 + 1 = -1.
\]

6. The two curves intersect in the \(x-y\) plane when \(x^2 + 3 = 2x + 6\) \(x^2 - 2x - 3 = 0\). \(x = -1\) and \(x = 3\). The area of \(R\) is \(\int_{-1}^{3} (2x + 6 - x^2 - 3) dx = \int_{-1}^{3} (3 + 2x - x^2) dx = \left[3x + x^2 - \frac{x^3}{3}\right]_{-1}^{3} = 9 + \frac{5}{3} = \frac{32}{3}\).

When \(R\) is transformed to \(R'\), the area of \(R'\) will be the area of \(R\) multiplied by \(4 \cdot 2 = 6\). The area of \(R'\) is therefore 64.

7. In order for two vectors to be linearly independent, they must not be scalar multiples of each other. Each pair of vectors are scalar multiples except for \(\begin{bmatrix}3 \\ 5 \\ 1\end{bmatrix}\) and \(\begin{bmatrix}6 \\ 10 \\ 3\end{bmatrix}\).

8. Let \(A = \begin{bmatrix}a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix}\), noting that \(A\) is symmetric. \(x^T Ax = \begin{bmatrix}x_1 & x_2\end{bmatrix} \begin{bmatrix}a_{11} & a_{12} \\ a_{12} & a_{22}\end{bmatrix} \begin{bmatrix}x_1 \\ x_2\end{bmatrix} = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 7x_1^2 - 4x_1x_2 + 5x_2^2\). Equating coefficients, we have \(2a_{12} = -4\) and hence \(a_{12} = -2\).
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9. Since the coefficients of \( f(x) \) are real numbers and exactly one root is real, the other two must be complex conjugates. Let the roots be \( r = a + bi \), \( s = a - bi \), and \( t \) where \( a \), \( b \), and \( t \) are real numbers. Then 
\[
\delta = \begin{vmatrix}
1 & a + bi & (a + bi)^2 \\
1 & a - bi & (a - bi)^2 \\
1 & t & t^2 
\end{vmatrix}
= \begin{vmatrix}
a + bi & a^2 - b^2 + 2abi \\
a - bi & a^2 - b^2 - 2abi \\
t & t^2 
\end{vmatrix}.
\]
First notice that \( \delta = 0 \) if and only if \( a = \pm t \), which is not the case. Also, notice that subtracting the second row from the first row will leave the determinant unchanged, so 
\[
\delta = \begin{vmatrix}
0 & 2bi & 4abi \\
1 & a - bi & a^2 - b^2 - 2abi \\
1 & t & t^2 
\end{vmatrix}
= -2bt^2i + 4abti + 2a^2bi - 2b^2i + 4ab^2 - 4a^2bi - 4ab^2
= -2bt^2i + 4abti - 2a^2bi - 2b^2i.
\]
This is a non-zero, purely imaginary number, so its square will be a negative real number. \( B \)

10. Notice that for any invertible matrix \( A \), \( \det(A A^{-1}) = 1 \). However, since all of the entries of both \( A \) and \( A^{-1} \) are integers, \( \det(A) \) and \( \det(A^{-1}) \) must also be integers. If the product of two integers is 1, then either both integers are 1 or both integers are \(-1\). \( B \)

11. \( \|v\| = \sqrt{\left(\frac{1}{3}\right)^2 + a^2} = 1 \). \( \frac{1}{9} + a^2 = 1 \). \( a^2 = \frac{8}{9} \). Since \( a \) is positive, we have \( a = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3} \). \( C \)

12. The trace of a matrix is the sum of the diagonal elements. The trace is therefore 9. \( D \)

13. Putting \( A \) is reduced row echelon form gives 
\[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}.
\]
The rank is the number of non-zero rows, which is 1. \( B \)

14. We want to find the vectors \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) such that 
\[
\begin{bmatrix}
1 & 2 & -1 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]
Then we must have 
\( x + 2y - z = 0 \), \( x = -2y + z \). We can take \( y \) and \( z \) as free variables. Setting \( y = 1 \) and \( z = 0 \), we get 
\( x = -2 \). Setting \( y = 0 \) and \( z = 1 \), we get \( x = 1 \). Then two linearly independent solutions are 
\[
\begin{bmatrix}
-2 \\
1 \\
0 
\end{bmatrix}
\quad \text{and} \quad 
\begin{bmatrix}
1 \\
0 \\
1 
\end{bmatrix}.
\]
The only choice that is not on the span of these two vectors is 
\[
\begin{bmatrix}
1 \\
1 \\
1 
\end{bmatrix}.
\]
This can be checked by noticing that 
\[
\begin{bmatrix}
1 & 2 & -1 \\
3 & 6 & -3 \\
-2 & -4 & 2 
\end{bmatrix}
\begin{bmatrix}
1 \\
1 \\
1 
\end{bmatrix} = \begin{bmatrix}
2 \\
6 \\
-4 
\end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]
\( D \)

15. \( B^T \) is a \( 2 \times 3 \) matrix, so \( B^T A \) is defined, and the product is a \( 2 \times 4 \) matrix. \( B \)

16. Notice that \( B = AA^T \). Therefore \( \det(B) = \det(AA^T) = \det(A)^2 = 4^2 = 16 \). \( D \)
17. Statement I is true. If $A$ is invertible, then the system $Ax = 0$ has only the trivial solution, and hence statement II is true. Since $0 = |A| = |A^T|$, statement III is also true. Statement IV is false.

Consider the matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ which is singular, but has trace 1. Three of the statements are true. C

18. Notice that $\cos(\theta) = \frac{\langle 1,1 \rangle \cdot (1+t,1-t)}{\|\langle 1,1 \rangle\| \cdot \|(1+t,1-t)\|} = \frac{2}{\sqrt{2} \sqrt{2t^2 + 2}} = \frac{1}{\sqrt{t^2 + 1}}$. Differentiating both sides with respect to $t$, $-\sin(\theta) \frac{d\theta}{dt} = -\frac{1}{2}(t^2+1)^{\frac{3}{2}} \cdot 2t = -t(t^2+1)^{\frac{3}{2}}$. Notice that when $t = 1$, $\cos(\theta) = \frac{\sqrt{2}}{2}$ and hence $\sin(\theta) = \frac{\sqrt{2}}{2}$ since $\theta$ is acute. Then when $t = 1$ $\frac{d\theta}{dt} = -\sqrt{2} \cdot (2)^{\frac{3}{2}} = \frac{1}{2}$. C

19. $D(x) = \begin{bmatrix} x \\ e^x \\ x+1 \end{bmatrix} = x(x+1) - e^x = x^2 + x - e^x$. $D'(x) = 2x + 1 - e^x$. $D'(0) = 0 + 1 - 0 = 0$. E

20. Let $x = \begin{bmatrix} x \\ y \end{bmatrix}$ and $x^2 + y^2 = 1$. Then $Ax = \begin{bmatrix} x \\ 2y \end{bmatrix}$. We would like to maximize $\sqrt{x^2 + 4y^2}$. $x^2 = 1 - y^2$, so we can maximize $\sqrt{1 + 3y^2}$. Realizing that $-1 \leq y \leq 1$, this is clearly maximized when either $y = 1$ or $y = -1$, and the maximum value is $\sqrt{4} = 2$. D

21. $A^3 = A \cdot A^2 = A(A + 2I_n) = A^2 + 2A = A + 2I_n + 2A = 3A + 2I_n$. C

22. $\|v \times i\| = \begin{vmatrix} 1 & x & y \\ 0 & y & x \\ 0 & 0 & 1 \end{vmatrix} = yj - xk = 1$. Therefore $x^2 + y^2 = 1$. $\|v\| = \sqrt{1^2 + x^2 + y^2} = \sqrt{2}$. B

23. Row reducing the augmented matrix for the system of equations,

$\begin{bmatrix} 2 & -6 & 8 \\ -3 & 9 & -12 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4 \\ -1 & 3 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 4 \\ 0 & 0 & 0 \end{bmatrix}$. Since the second row is all zeros, the system of equations can be represented as a single equation in two variables (this is because the second equation is a scalar multiple of the first). Since there are more variables than there are linearly independent equations, there are infinitely many solutions. D

24. $\|\langle t,2+t\rangle\| = \sqrt{t^2 + (2+t)^2} = \sqrt{2t^2 + 4t + 4}$. For simplicity, we can find the $t$ that will minimize $2t^2 + 4t + 4$, realizing that this value of $t$ will also minimize the square root. Taking the derivative and setting it equal to 0, we get $4t + 4 = 0$, and hence $t = -1$. Notice that the second derivative is $4 > 0$, and hence this does indeed lead to a minimum. E

25. By plotting the vectors in the $x$-$y$ plane and using the right-hand rule, it can be seen that only the set of vectors $u = \langle 5,1,0 \rangle$ and $v = \langle 2,3,0 \rangle$ will satisfy $u \times v = \langle 0,0,a \rangle$, where $a > 0$. A

26. Let $x$ be the price of an apple and $y$ be the price of an orange. We then want to solve for $y$ where $4x + 7y = 43$ and $3x + y = 11$. Using Cramer's Rule, $x = \begin{vmatrix} 4 & 43 \\ 3 & 11 \end{vmatrix} = \begin{vmatrix} 4 & 43 \\ 4 & 7 \end{vmatrix} = \begin{vmatrix} 4 & 132 \\ 4 & 21 \end{vmatrix} = \frac{44 - 132}{-17} = \frac{-88}{-17} = 5$. D
27. \[ \begin{bmatrix} -1 & 2 \\ 8 & -1 \end{bmatrix} \mathbf{x} = \lambda \mathbf{x} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \mathbf{x} \]. Then \[ \begin{bmatrix} -1 - \lambda & 2 \\ 8 & -1 - \lambda \end{bmatrix} \mathbf{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \]. Such a system of equations can only have a non-zero solution if the matrix \[ \begin{bmatrix} -1 - \lambda & 2 \\ 8 & -1 - \lambda \end{bmatrix} \] is singular, that is, \[ \begin{vmatrix} -1 - \lambda & 2 \\ 8 & -1 - \lambda \end{vmatrix} = 0 \].

Then \( \lambda^2 + 2\lambda - 15 = 0 \) and \( \lambda = 3 \) or \( \lambda = -5 \). **C**

28. The cofactor is \( (-1)^{2+3} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = -3 \). **B**

29. \( |A| = |A^T| = |-A| = (-1)^{11} = -|A| \). As a result, we must have that \(|A| = 0 \). **A**

30. \( \|\langle 1, 2, 3 \rangle - 2 \cdot \langle -1, 4, 3 \rangle\| = \|\langle 3, -6, -3 \rangle\| = \sqrt{3^2 + (-6)^2 + 3^2} = \sqrt{9 + 36 + 9} = \sqrt{54} = 3\sqrt{6} \). **C**