2010 Nationals Mu Sequences and Series Solutions

1. I) False. Counter Example: $S_n = \frac{1}{n}$

   II) True

   III) False. The converse is true.

   IV) True. If $\sum S_n$ diverges towards $\pm\infty$ then $\lim_{n\to\infty} \frac{1}{\sum S_n} = 0$.

2. Truncated to the x^2 term

   $$(x+1)^{-1/3} = \binom{-1/3}{0} x^0 (1)^{-1/3} + \binom{-1/3}{1} x^1 (1)^{-4/3} + \binom{-1/3}{2} x^2 (1)^{-7/3}.$$

   To deal with negative combinations we consider a revised version such that $\binom{a}{b} = \frac{a(a-1)(a-2)\cdots(a-b+1)}{b!}$.

   Thus we have

   $$(.2+1)^{-1/3} = (1)(.2)^0 (1)^{-1/3} + \left(-1/3\right)(.2)^1 (1)^{-4/3} + \left(2/9\right)(.2)^2 (1)^{-7/3} = \frac{212}{225}. \text{ D}$$

3. $$\prod_{n=2}^{\infty} \sqrt{\left(1-\frac{1}{n^2}\right)} = \sqrt{\prod_{n=2}^{\infty}\left(1-\frac{1}{n^2}\right)} = \sqrt{\prod_{n=2}^{\infty}\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)} = \sqrt{\prod_{n=2}^{\infty}\left(\frac{n-1}{n}\right)\left(\frac{n+1}{n}\right)} =$$
   $$\lim_{n\to\infty} \sqrt{\frac{1}{2}\left(\frac{n+1}{n}\right)} = \frac{\sqrt{2}}{2}. \text{ B}$$

4. Integral Test: $\int_0^{\infty} (2n+1)^{-1/2} = \lim_{n\to\infty} 2(2n+1)^{1/2}\Big|_0^n = \infty \to \textit{Diverges}$. C

5. Ratio Test: $\lim_{n\to\infty} \left| \frac{\ln(n+1)}{(n+1)!} \frac{n!}{\ln(n)} \right| = \lim_{n\to\infty} \frac{1}{n+1} = 0 \to$ Absolute Convergence. A

6. Alternating Series Test: $\lim_{n\to\infty} \left(\frac{n!}{4^n}\right) = \infty \to$ Divergence. C

7. Alternating Series Test: $\lim_{n\to\infty} \frac{n^2}{n^3+1} = 0$, $\frac{n^2}{n^3+1} > \frac{(n+1)^2}{(n+1)^3+1} \to$ Convergence.

   Checking the convergence of $\sum \frac{n^2}{n^3+1}$ through the Integral test reveals that it diverges. Thus the series converges conditionally. B

8. Comparison Test: $\frac{\ln(n)}{\sqrt{n}} \ge \frac{1}{\sqrt{n}}$. Since $\sum \frac{1}{\sqrt{n}}$ diverges then $\sum \frac{\ln(n)}{\sqrt{n}}$ diverges. C

9. $\sum_{n=0}^{\infty}\frac{n^2}{n!}=\sum_{n=0}^{\infty}\frac{n^2x^2}{n!}$ evaluated at x = 1. To get $\sum_{n=0}^{\infty}\frac{n^2x^2}{n!}$ consider $y=e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.

Thus we have $xy'=xe^x=\sum_{n=0}^{\infty}\frac{nx^n}{n!}$. Differentiating again produces

$xe^x+e^x=\sum_{n=1}^{\infty}\frac{n^2x^{n-1}}{n!}$. Evaluated at x = 1 gives $2e$. C

10. The first 100 terms consist of the first 50 terms from R and the first 50 terms from P. The remainder from a geometric series after the first n terms is $|R_n|=\left|\frac{ar^n}{1-r}\right|$. For alternating series the remainder is the next term, so in this case it would be P_{51}. The remainder is thus

$$R_{51}+P_{51}=\left|\frac{\left(\frac{1}{2}\right)^{50}}{-\frac{1}{2}}\right|+\frac{1}{99}=\left(\frac{1}{2}\right)^{49}+\frac{1}{99}.\text{ A}$$

11. $f(x)=\arctan(\sin x)\rightarrow f'(x)=\frac{\cos x}{1+\sin^2 x}\rightarrow f'\left(\frac{\pi}{4}\right)=\frac{\frac{\sqrt{2}}{2}}{\frac{3}{2}}=\frac{\sqrt{2}}{3}$. B

12. To put $f(x)$ in terms of $(x-3)$ consider the Taylor Series for the function centered at x = 3. We have $f(3)=-2$, $f'(3)=0$, $f''(3)=2$. Therefore

$f(x)=-2+0(x-3)+\frac{2(x-3)^2}{2!}=(x-3)^2-2$.

$\int_3^{3.2}f(x)dx=\int_3^{3.2}\left((x-3)-2\right)dx=\frac{(x-3)^3}{3}-2x\Big|_3^{3.2}=\frac{.2^3}{3}-.4=-\frac{149}{375}$. C

13. The series reduces to $e^{-i\pi}=\cos(-\pi)+i\sin(-\pi)=-1$. A

14. Factoring out pi we have $\frac{1}{\pi}\left(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots\right)$. The series in question is the MacLaurin series for $\arctan x$ at $x=1$. Thus the series evaluates to $\frac{1}{\pi}\bullet\frac{\pi}{4}=\frac{1}{4}$. C

15. $\lim_{n\to\infty}\left|\frac{(n+1)(x+3)^{n+1}}{(x-1)^{2n+2}e^{(n+1)^2}}\frac{(x-1)^{2n}e^{n^2}}{n(x+3)^n}\right|=\lim_{n\to\infty}\left|\frac{(x+3)}{(x-1)^2e^{2n+1}}\right|=0$.

Thus the series converges for all values except x = 1 which would cause the series to be undefined. The interval of convergence is $(-\infty,1)\cap(1,\infty)$. C

16. $\sum_{j=0}^{\infty}\frac{1}{9j^2-3j-2}=\frac{1}{3}\sum_{j=0}^{\infty}\left(\frac{1}{3j-2}-\frac{1}{3j+1}\right)=\frac{1}{3}\left(-\frac{1}{2}\right)=-\frac{1}{6}$. A

17. $N=\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ which is a well known fact proven by Euler. Since $\cos=\sin(90-x)$ we can reduce $\prod_{i=1}^{8}\tan(10i)=\frac{\sin 10\sin 20\cdots\sin 80}{\cos 10\cos 20\cdots\cos 80}$ to 1. Thus the summation is really $\sum_{j=1}^{6}j=21$. D

18. When dealing with continued fractions we multiply by an appropriate power of the base we are working in. For example, consider $x=.\overline{3}$ in base 10. Thus we have $10x=3.\overline{3}$. Subtracting we get $9x=3$ and thus $x=\frac{1}{3}$. For the given problem we are working in base 9 thus we have
$x=.\overline{481}$ and $9^3x=729x=481.\overline{481}$. Subtracting yields
$728x=481\to x=\frac{481}{728}=\frac{37}{56}$. C

19. This is a Riemann sum which reduces to $\int_0^1\sqrt{x}dx=\frac{2}{3}x^{3/2}\Big|_0^1=\frac{2}{3}$. C

20. $a^2=9+12\sqrt{2}+8$ and $b^2=9-12\sqrt{2}+8$. Thus $R_2=\frac{1}{2}(34)=17$ which in base 2 is 10001_2. The sum of the digits in base 2 is $1+1=2=10_2$. B

21. $\sum_{n=1}^{\infty}\frac{(\pi/2)^{2n-1}}{(2n-1)!}=\sin\frac{\pi}{2}=1$, $\sum_{n=0}^{\infty}\frac{(-2)^n}{n!}=e^{-2}$, $\sum_{n=0}^{\infty}\frac{4^n}{n!}=e^4$. The product is thus $1(e^{-2})(e^4)=e^2$. B

22. $f(x)=\frac{3x-1}{x^2-1}=\frac{2}{x+1}+\frac{1}{x-1}$

$\frac{2}{x+1}=\frac{2}{1-(-x)}=\sum_{n=0}^{\infty}2(-x)^n$ and $\frac{1}{x-1}=\frac{-1}{1-x}=-\sum_{n=0}^{\infty}x^n$ for $|x|<1$

Thus $\frac{3x-1}{x^2-1}=\sum_{n=0}^{\infty}\left[2(-1)^n-1\right]x^n$. B

23. $\lim_{n\to\infty}\left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = |(n+1)x| = \infty$. This converges only when x = 0. Thus the radius of convergence is 0. A

24. $xe^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots$ integrating term-wise gives

$$\int_0^1 xe^x\,dx = \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8} + \frac{x^5}{30} + \ldots\Big|_0^1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{30} + \ldots = xe^x - e^x\Big|_0^1 = 1$$

Looking at terms of $\sum_{n=0}^{\infty}\frac{1}{(n+2)n!} = \frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{30} + \ldots$ shows that they are equivalent. Thus the summation is equal to 1. B

25. Sum = $\frac{a}{1-r}$ where a= first term and r=ratio. Thus we have

sum = $\frac{i+1}{1-r} = 3i+4$. Solving this for r gives $r = \frac{2i+3}{3i+4} = \frac{2i+3}{3i+4}\frac{3-i}{3-i} = \frac{i+18}{25}$. C

26. $S_5 = a + 4n = 56$ and $S_{25} = a + 24n = 105$. Solving for n gives $n = \frac{49}{20}$.

Plugging this back in and solving gives $a + 4\left(\frac{49}{20}\right) = 56 \to a = \frac{231}{5}$.

Thus $S_{21} = \frac{231}{5} + 20\left(\frac{49}{20}\right) = \frac{476}{5}$. E

27. Let the limit be $y = \sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2\sqrt{2\ldots}}}}}} = \sqrt{2y}$. Squaring both sides and grouping yields $y^2 - 2y = 0 \to y = 0, y = 2$. Since the square root is greater than zero the answer is 2. C

28. The sum of the squares first 10 positive odd integers is simply the sum of the squares first 20 integers minus the sum of the squares of the first 10 even numbers. The sum of the squares of first 20 integers is

$\sum_{i=1}^{20} i^2 = \frac{(20)(21)(41)}{6} = 2870$. To find the sum of the squares of the first 15 odd integers consider this $2^2 + 4^2 + \cdots + 20^2 = 2^2(1^2 + 2^2 + \cdots + 10^2)$. Thus the sum of the squares of the first 10 even integers is $4\sum_{i=1}^{10} i^2 = \frac{4(10)(11)(21)}{6} = 1540$.

The answer is then $2870 - 1540 = 1330$. D

29. For $n > 4$, n! ends in a zero. Thus the last digit of $\sum_{n=0}^{\infty} n!$ is equal to the last digit of $\sum_{n=0}^{4} n!$. This gives 1+1+2+6+24=34. The last digit is 4. B

30. This is equivalent to the sum of two geometric series, one with ratio ½ and one with ratio 1/3. The sum is $\frac{1}{1-1/2} + \frac{1/3}{1-1/3} = 5/2$. E