1. There are two ways to solve this. One could square each number by hand and check. However, the shorter method would be to work in modulo 10. For D, this would imply that \(6^2 + 8^2 \equiv 2^2\) which is false. The other choices pass this first test, hence, the answer is D.

2. 24! has factors of 5, 10, 15, 20 which all have a factor of 5; hence 24! is divisible by \(10^4\) but not \(10^5\). 25! has factors 25, 20, 15, 10, 5. Since 25 = \(5^2\), that means 25! has 6 factors of 5. Hence there is no number whose factorial meets the given conditions and the correct answer is E.

3. 224 = \(32\times7 = 2^5 \times 7\) so any number of the form \(2^x \times 7^y\) is a factor of 224, where \(0 \leq x \leq 5\) and \(0 \leq y \leq 1\). So there are two choices for y, 6 choices for x. 12 factors total, but 224 is not a proper divisor of itself, so 11 proper divisors. C.

4. Any perfect square must be \(\equiv 0\) or \(\equiv 1\) (mod 4), so B is true. Since \(k = 4n + 3\), \(k\) is odd, so A is true. C is not true; the correct combination would be \(k \equiv 23\) (mod 28) so the correct answer is D.

5. Clearly \(\gcd(11,37) = 1\) since both are prime. Using the Euclidean algorithm, we get that for \(37a + 11b = \gcd(11,37) = 1\), \(a = 3\) and \(b = -10\). We could multiply this by 3 and use the pair \((9, -10)\) but 9 is not a given answer. Instead, note that an intermediate step of the Euclidean algorithm yields the pair \((-2, 7)\). B.

6. \(\varphi(36) = \varphi(2^2 \times 3^2) = \varphi(2^2)\varphi(3^2) = (2^1)(2^1)(3^1)(3^1) = 2 \times 1 \times 3 \times 2 = 12\) A.

7. \(\varphi(7) = 6, \varphi(13) = 12, \varphi(19) = 18, \varphi(35) = 24\). Other solutions exist; these are the smallest x. 14 and 26 are nontotient. B.

8. A is infinite (known since Euclid, at least). B is infinite; since it is not specified that they be primitive, we could take for example all integral multiples of \((3,4,5)\). D is infinite, if we consider only polygons with \(2^n\) sides by constructing a square, and bisecting central angles. C may be infinite, but this is unknown (Twin Prime Conjecture) so the answer is C.
9. The sum of the digits 1+5+1+4+7 = 18, so 15147 is divisible by 9; 15147 = 9*1683. Sum of
digits of 1683 is 18, so divide again by 9, =3^4 * 187. 187 = 11*17 (if this isn't immediately
clear, try the mod 11 test of 1+8-7 = 0) so 15147 = 3^4 * 11 * 17. 3+11+17 = 31  D

10. If \( k \) is prime, then \( k^2 \) has two proper divisors, 1 and \( k \). If \( k \) is a perfect square, then \( k^2 \) has an odd
number of proper divisors. If \( k = p^3 \), then \( k^2 = p^6 \) has proper divisors 1, \( p \), \ldots through \( p^5 \), so
there are 6 proper divisors. If \( k \) has 4 proper divisors, then \( k^2 \) has at least 8 proper divisors, just
counting the divisors of \( k \) and their squares. The answer is C

11. 8 cannot be expressed as the sum of two cubes (as defined in the start of the test). \( 72 = 4^3 + 2^3 \)
which is only one sum. \( 1216 = 10^3 + 6^3 \), similarly only one way to do so. \( 1729 = 1^3 + 12^3 =
10^3 + 9^3 \) so the correct answer is B

12. By Fermat's Little Theorem, 23 is prime so \( 11^{22} \) is congruent to 1. A

13. 191 is prime, so the \( N = 1 \) and \( M = 191*214 = 40874 \). \( M - N = 40873 \)  D

14. There are 11 partitions. In a slightly compressed list form, they are: 6, 15, 114, 24, 33, 1113,
123, 222, 1122, 11112, 111111  E

15. I may be true in certain cases (for example, 15 divides 14! since 5 and 3 and both divisors of
14!) but is not true in general (for example, whenever \( n+1 \) is prime). II is true because \( n+1 \)
includes (but is not limited to) any partition of the form \( \{1, X\} \) where \( X \) is any partition of \( n \).
III is the Bertrand-Chebyshev Theorem, which is true only for \( n \geq 2 \), a condition we do have
here. II and III are correct, so the answer is E

16. There are \( x = 7 \) partitions of 5 and \( y = 5 \) partitions of 4. \( 7 \pmod{5} = 2 \)  B

17. Solving the second and third congruences gives \( x \equiv 56 \pmod{143} \); adding in the first
congruence we get \( x \equiv 485 \pmod{1001} \). 1001 / 485 = 2.060, so the closest integer is 2  D

18. By Fermat's Little Theorem, \( 13^{30} \equiv 1 \pmod{31} \), so \( 13^{31} \equiv 13 \pmod{31} \)  C
19. \( A = \gcd(7 \times 12, 3 \times 12) = 12; \ B = \text{lcm}(14, 14 \times 7) = 98; \ C = 120 - 5 = 115; \ D \equiv 4 \pmod{7}, \) but D is not used in the expression. The expression is thus \( (12 + 98)(115 - 12) = (110)(103) = 11000 + 330 = 11330 \)  

20. The golden ratio \( \phi = \frac{1+\sqrt{5}}{2}, \) the solution to the equation \( x^2 - x - 1 = 0 \)  

21. Algebraic numbers are closed under addition and multiplication, so I and III are algebraic. Logarithms are not necessarily algebraic, and neither are exponents.  

22. If we have a polynomial in the rationals, we can turn it into a polynomial with integer coefficients by multiplying by the LCM of all denominators of fractional coefficients. Hence, \( \alpha \) is certainly algebraic since it will also be a solution to this integer polynomial.  

23. Long division could be used here, but the shortcut is to sum up the digits; if the sum of the digits of a number is divisible by 9 then the number itself is. Number A sums to 40. Number B sums to 61. Number C sums to 39. Number D sums to 37. None of them are divisible by 9, so the correct answer is E  

24. \( 8741 = 2(4096) + 1(512) + 0(64) + 4(8) + 5(1) \) so 21045  

25. \( 135 = (3^3)(5). \) Since both prime factors are odd, any factor of 135 is odd, so probability of choosing an odd factor is = 1  

26. There are 25 prime numbers less than 100; however Sven will not step on the number 3, since 2 is prime he skips over 3. Thus he skips 24 steps, and stands on the other 76.  

27. Sven's path downstairs is the following (primes in bold): 100, 99, 98, 97, 87, 86, 85, 84, 83, 73, 64, 63, 62, 61, 53, 45, 44, 43, 36, 35, 34, 33, 32, 31, 25, 24, 23, 18, 17, 12, 11, 7, 4, 3, 1. He touches 12 prime-numbered steps.  

28. Reals and positive integers are closed under exponentiation; this should be obvious. The set in IV is also obviously closed. Rationals are not, because for example \( 2^{1/2} = \sqrt{2} \) is not rational
29. The Fibonacci sequence is, in part: 0, 1, 2, 3, 5, 8, … The sum of the first 6 is $0+1+1+2+3+5 = 12$ (Or a shortcut, the sum is one less than the next Fibonacci number, in this case $5+8=13$).

30. Gaussian integers (and all complex numbers in general) retain commutativity and associativity of multiplication. Unique factorization holds for Gaussian integers; non-unit elements factor uniquely into products of irreducible elements. Not that a counterexample such as $10 = (2)(5) = (3+i)(3-i)$ is not valid, since $2 = (1+i)(1-i)$ so 2 is not irreducible in $\mathbb{Z}[i]$. $\mathbb{Z}$ is totally ordered, whereas any ordering of $\mathbb{Z}[i]$ does not preserve the relevant arithmetic.