Answers:

Relay 1
1. 11
2. 82
3. 3321
4. 58
5. 21
6. 15

Relay 2
1. 9
2. 5
3. 18
4. 27
5. 30
6. 1369

Relay 3
1. 7
2. 112
3. 34
4. 54
5. 19
6. 19

Relay 4
1. 150
2. 90
3. 28
4. 7
5. 10
6. 182

Relay 5
1. 19958400
2. 416
3. 10
4. 54
5. 19
6. 210

Solutions:

Relay 1

1. $13(16x-17y=44) \Rightarrow 208x-221y=572$ and $16(13x+5y=111)$
$\Rightarrow 208x+80y=1776$. Therefore, $-301y=-1204 \Rightarrow y=4 \Rightarrow x=7$. So $7+4=11$.

2. $180-\theta=10+11(90-\theta) \Rightarrow 180-\theta=10+990-11\theta \Rightarrow 10\theta=820 \Rightarrow \theta=82$

3. $\binom{82}{2}=3321$

4. $57^2=3249$ and $58^2=3364$, so the answer is 58

5. The only solution to $x^2+y^2=58$ with $x<y$ is $(3,7)$. $3\cdot 7=21$

6. $x(21-x)=9\cdot 10=90 \Rightarrow x^2-21x+90=0 \Rightarrow (x-6)(x-15)=0$. The larger segment has length 15.

Relay 2

1. $\left(\frac{4}{12}\right)^2=\left(\frac{1}{3}\right)^2=\frac{1}{9}$, so the answer is 9

2. $x!$ is divisible by 9 for all $x\geq 6$, so the sum of the digits of these numbers is also divisible by 9. $5!=120$, which is not divisible by 9, so the answer is 5

3. The graph of $y=-5|x-1|+3$ intersects the x-axis at $\frac{2}{5}$ and $\frac{8}{5}$, and the bounded area is a triangle with height 3 and base $\frac{6}{5}$. The area is $\frac{1}{2}\left(\frac{6}{5}\right)(3)=\frac{9}{5}$, and $10\left(\frac{9}{5}\right)=18$.

4. Let $R=$ the larger radius and $r=$ the smaller radius. Therefore,
$18^2+(R-r)^2=(R+r)^2$. Since $r=3$, $324+R^2-6R+9=R^2+6R+9 \Rightarrow 324=12R$
$\Rightarrow 27=R$.

5. Since $27=3^3$, the only possible common ratios are powers of 3. Since the exponent 3 is prime, the only possible common ratios are $3^1=3$ and $3^3=27$. $3+27=30$

6. $\frac{c}{a}=e=\frac{35}{37}\Rightarrow c=\frac{35}{37}a$ and $\frac{2b^2}{a}=144\Rightarrow b^2=72a$, so because this is an ellipse, $a^2-72a=\left(\frac{35}{37}a\right)^2\Rightarrow 0=\frac{144}{1369}a^2-72a=a\left(\frac{144}{1369}a-72\right)\Rightarrow a=\frac{1369}{2}\Rightarrow 2a=1369$, so the major axis has length 1369.

Relay 3

1. $f(x)=\frac{(3x+5)(2x-1)}{(4x-7)(2x-1)}$, so the only vertical asymptote is when $x=7/4$. $4(7/4)=7$

2. $14^2-4\cdot 3\cdot 7=196-84=112$

3. The other two angles add to $180-112=68^\circ$, and since the two angles are equal, they are each of measure 34°

4. The only positive integral factors are 1, 2, 17, and 34, and $1+2+17+34=54$

5. The sides are of length 54, 36, and 45. The angle bisector to the side of length 36 is of length $\sqrt{54\cdot 45\left(1-\frac{36^2}{99^2}\right)}=\sqrt{54\cdot 45\left(\frac{105}{121}\right)}=\frac{9\cdot 5\cdot 3\sqrt{14}}{11}=\frac{135\sqrt{14}}{11}$.
$14\cdot 11-135=154-135=19$

6. $19\cos\left(19-1-\frac{360}{19+1}\right)^\circ=19\cos(18-18)^\circ=19\cos 0^\circ=19$

Relay 4

1. The area is $49\pi\approx 49(3.14)=153.86$, so the answer is 150.

2. A convex 15-gon has a total of $\frac{15\cdot 12}{2}=90$ diagonals.

3. The society has 9 members. It takes 2 of the other 8 members to form a handshake with Bob, so the number of handshakes involving Bob is $\binom{8}{2}=28$.

4. $\frac{x(x+1)}{2}=28$, and the positive integer solution to this is $x=7$

5. $\begin{bmatrix}3 & 2\\ k & 2\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=7\begin{bmatrix}x\\ y\end{bmatrix}\Rightarrow 3x+2y=7x$ and $kx+2y=7y\Rightarrow 2y=4x$ and $kx=5y\Rightarrow y=2x$ and $kx=5y\Rightarrow kx=10x$, so $k=10$

6. In this right triangle, $(c-b)(c+b)=c^2-b^2=20^2=400$, and the only way to have this factorization consist only of integers is if $c-b$ and $c+b$ are both even or both odd. Since 400 is even, at least one factor must be even, meaning both $c-b$ and $c+b$ must be even. The only factorizations that fit this are $2\cdot 200$, $4\cdot 100$, $8\cdot 50$, or $10\cdot 40$, and setting the smaller of these factors equal to $c-b$ and the larger equal to $c+b$, the corresponding Pythagorean triples are $(20,99,101)$, $(20,48,52)$, $(20,21,29)$, and $(15,20,25)$. This last triple doesn't have 20 as its smallest leg length, but the other three do, so the sum of all possible hypotenuse lengths for the triangle is $101+52+29=182$.

Relay 5

1. Since there are 3 As, 2 Ts, and 2 Hs, the number of distinguishable permutations is $\frac{12!}{3!2!2!}=19958400$.

2. $19958400/100=199584$, and to get to a number divisible by 1000, you must add 6, then 10, then 400, so the answer is $6+10+400=416$.

3. Every four terms of $\sum_{n=1}^{416} ni^n$ adds up to $i-2-3i+4=2-2i$, so the sum is $104(2-2i)=208-208i$, so the answer is $2+0+8=10$.

4. The sequence is 10, 2, 5, 26, 41, 18, 66, 73, 59, 107, 51, 27, 54, 42, 21, 6, 37,..., and 37 yields the same successor as 73. Therefore, this sequence has 8 initial terms, then

the next 9 that repeat over and over. $2011 = 8 + 9 \cdot 222 + 5$, so the 2011th term is the same as the 5th term in the repeating cycle, which is 54.

5. The distance is $\dfrac{\left|3(-7)-4(5)-54\right|}{\sqrt{3^2+(-4)^2}} = \dfrac{\left|-21-20-54\right|}{5} = \dfrac{95}{5} = 19$.

6. The triangle has sides of length 20, 21, and 29, which is a right triangle, and therefore has area $\frac{1}{2}(20)(21) = 210$.