Answers:

1. B
2. A
3. C
4. D
5. D
6. A
7. B
8. B
9. A
10. E
11. A
12. B
13. B
14. D
15. A
16. C
17. D
18. B
19. A
20. D
21. C
22. E
23. C
24. C
25. D
26. A
27. B
28. D
29. D
30. C

Solutions:

1. $\dfrac{3(4)^2\sqrt{3}}{2}=24\sqrt{3}$

2. $\dfrac{4}{3}\pi(6)^3=288\pi$

3. $\int_0^2 3e^{-3x}dx=-e^{-3x}\Big|_0^2=1-e^{-6}$

4. The two graphs intersect at the points $(5,17)$ and $(-2,3)$, and the area enclosed is

$$\int_{-2}^{5}\left((2x+7)-\left(x^2-x-3\right)\right)dx=\int_{-2}^{5}\left(10+3x-x^2\right)dx=\left(10x+\frac{3}{2}x^2-\frac{1}{3}x^3\right)\Bigg|_{-2}^{5}=50+\frac{75}{2}-\frac{125}{3}$$
$$+20-6-\frac{8}{3}=\frac{343}{6}$$

5. $\pi\int_0^{0.5}\left(\dfrac{1}{\sqrt[4]{1-x^2}}\right)^2dx=\pi\int_0^{0.5}\dfrac{1}{\sqrt{1-x^2}}dx=\pi\left(\sin^{-1}x\right)\Big|_0^{0.5}=\pi\left(\dfrac{\pi}{6}\right)=\dfrac{\pi^2}{6}$

6. $\pi\int_0^4\left(\left(4\sqrt{x}\right)^2-(2x)^2\right)dx=\pi\int_0^4\left(16x-4x^2\right)dx=\pi\left(8x^2-\dfrac{4}{3}x^3\right)\Bigg|_0^4=\pi\left(128-\dfrac{256}{3}\right)=\dfrac{128\pi}{3}$

7. $2\pi\int_1^5 x\cdot\dfrac{5}{x}dx=2\pi\int_1^5 5dx=2\pi\cdot 5(5-1)=40\pi$

8. $\int_0^{2/3}(2-3x)dx+\int_{2/3}^{3}(3x-2)dx=\left(2x-\dfrac{3}{2}x^2\right)\Bigg|_0^{2/3}+\left(\dfrac{3}{2}x^2-2x\right)\Bigg|_{2/3}^{3}=\dfrac{4}{3}-\dfrac{2}{3}+\dfrac{27}{2}-6-\dfrac{2}{3}+\dfrac{4}{3}$

$=\dfrac{53}{6}$, so $m+n=53+6=59$

9. $\int_a^b f(x)dx$ will have the same units as the product of $f(x)$'s and x's units, so the units are $\dfrac{\text{ft}^3}{\text{sec}^2}\cdot\text{ft}\cdot\text{sec}=\dfrac{\text{ft}^4}{\text{sec}}$

10. None of the answer choices A, B, or C are necessarily true as each of them assume that f is nonnegative on the interval.

11. For later use, the area of a parabolic sector is $\int_{-b/2}^{b/2}\left(h-\frac{4hx^2}{b^2}\right)dx=2\left(hx-\frac{4hx^3}{3b^2}\right)\Big|_0^{b/2}$

$=2\left(\frac{hb}{2}-\frac{hb}{6}\right)=\frac{2}{3}bh$, so the answer is $\frac{2}{3}(6)(4)=16$.

12. $\int_0^{\pi}\sin x dx=(-\cos x)\big|_0^{\pi}=1+1=2$, so $\frac{2}{3}=\int_0^{a}\sin x dx=(-\cos x)\big|_0^{a}=1-\cos a\Rightarrow\cos a=\frac{1}{3}$.

Because of symmetry of this region, $\cos b=-\frac{1}{3}$, and thus $\sin a=\sin b=\frac{2\sqrt{2}}{3}$.

Therefore, $\sin(b-a)=\sin b\cos a-\cos b\sin a=\frac{2\sqrt{2}}{3}\left(\frac{1}{3}+\frac{1}{3}\right)=\frac{4\sqrt{2}}{9}$.

13. $\frac{\sqrt{3}}{4}\int_0^5(5-x)^2dx=\frac{\sqrt{3}}{4}\cdot-\frac{1}{3}(5-x)^3\Big|_0^5=\frac{125\sqrt{3}}{12}$

14. The integrand is a cubic, and Simpson's Rule provides exact values for quadratic and cubic functions. Thus, the "approximation" is $\int_0^4\left(x^3-4x+2\right)dx=\left(\frac{1}{4}x^4-2x^2+2x\right)\Big|_0^4$

$=64-32+8=40$.

15. This is the probability of selecting a point inside the circle centered at the origin with radius 2 out of all points inside the square with side length 8 centered at the origin. Therefore, the probability is $\frac{\pi(2)^2}{8^2}=\frac{\pi}{16}$.

16. The area under one arc of the sine graph is $\int_0^{\pi}\sin x dx=(-\cos x)\big|_0^{\pi}=1+1=2$, and for each whole number n, $f(n)$ is the area under $2n$ arcs of the sine graph. Therefore, $f(n)=2n(2)=4n$. $\sum_{n=1}^{100}f(n)=\sum_{n=1}^{100}4n=4\cdot\frac{100\cdot101}{2}=20200$.

17. Since $y'=5x^4$, $y'\big|_{x=5}=5^5=3125$. Therefore, the tangent line has equation $y-3125=3125(x-5)$, which has intercepts at the points $(0,-12500)$ and $(4,0)$, so the area enclosed by this line and the axes is $\frac{1}{2}(12500)(4)=25000$.

18. $\int_0^5 g(x)dx=1(0+1+2+3+4)=10$

19. The first graph forms a triangle with vertices at the points $(6,6)$, $(9,3)$, and $(9,9)$. The second graph forms the right half of a circle centered at $(9,6)$ with radius 3. The area enclosed by these two graphs is $\frac{1}{2}(3)(6)+\frac{1}{2}\pi(3)^2=9+\frac{9\pi}{2}$. By the Theorem of Pappus, the volume is $2\pi(6)\left(9+\frac{9\pi}{2}\right)=54\pi(\pi+2)$.

<u>OR</u>

Using the washer method and the formula for the volume enclosed by a torus, the volume is $\pi\int_6^9\left(x^2-(12-x)^2\right)dx+\frac{1}{2}\cdot 2\pi^2(6)(3)^2=\pi\int_6^9(24x-144)dx+54\pi^2$

$=\pi\left(12x^2-144x\right)\Big|_6^9+54\pi^2=\pi(972-1296-432+864)+54\pi^2=108\pi+54\pi^2$

$=54\pi(\pi+2)$

20. The graph, in the first quadrant and relevant to the bounded region, consists of line segments connecting the points $(0,6)$, $(1,3)$, $(2,2)$, $(3,3)$, and $(4,6)$, so interpreting the integral as an area, $\int_0^4\left(|x-1|+|x-2|+|x-3|\right)dx=\frac{1}{2}(1)(2)\left((6+3)+(3+2)\right)=14$.

21. Note that $f(0)=2$ and $f(4)=78$. Since graphs of functions and their inverses are mirror images through the line $y=x$, we get that $\int_0^4 f(x)dx+\int_2^{78}h(y)dy=78\cdot 4$ $=312$. Therefore, $\int_2^{78}h(y)dy=312-\int_0^4\left(x^3+3x+2\right)dx=312-\left(\frac{1}{4}x^4+\frac{3}{2}x^2+2x\right)\Big|_0^4$ $=312-(64+24+8)=216$.

22. Setting $b=0$ creates a cubic function with positive leading coefficient, so there is no maximum value.

23. The $\sqrt{4-\frac{(x-10)^2}{4}}$ portion of the integral describes the top half of an ellipse centered at $(10,0)$ with major and minor radius of 4 and 2, respectively. The area of the semi-ellipse is $\frac{\pi ab}{2}=\frac{\pi(4)(2)}{2}=4\pi$. The integral can be interpreted as the volume obtained when the semi-ellipse is revolved about the line $x=3$. By the Theorem of Pappus, the volume is $2\pi RA=2\pi(10-3)(4\pi)=56\pi^2$.

<u>OR</u>

$2\pi\int_{6}^{14}(x-3)\left(4-\frac{(x-10)^2}{4}\right)^{1/2}dx=\pi\int_{6}^{14}(x-3)\sqrt{16-(x-10)^2}\,dx$. Making the substitutions $u=x-10$ and $du=dx$, the integral becomes $\pi\int_{-4}^{4}(u+7)\sqrt{16-u^2}\,du$, which can be split up as $\pi\left(\int_{-4}^{4}u\sqrt{16-u^2}\,du+\int_{-4}^{4}7\sqrt{16-u^2}\,du\right)$. Because the integrand for the first integral is odd, the first integral is equal to 0. Interpreting the second integral in terms of area, the second integral is 7 times the area under the semicircle centered at the origin with radius 4. Therefore, the integral is equal to $\pi\left(0+7\left(\frac{1}{2}\right)(\pi)(4)^2\right)=56\pi^2$.

24. The frustum can be modeled by revolving the region bounded by the graphs of $y=\frac{1}{4}x+5$ and $y=0$ on the interval $0\le x\le 12$ about the x-axis. At a general water level h, the volume is $V=\pi\int_{0}^{h}\left(\frac{1}{4}x+5\right)^2dx$. Differentiating with respect to time we get $\frac{dV}{dt}=\pi\left(\frac{1}{4}h+5\right)^2\frac{dh}{dt}$. Plugging in the given information yields the equation $\pi=\pi\left(\frac{1}{4}(6)+5\right)^2\frac{dh}{dt}\Rightarrow\frac{dh}{dt}=\frac{4}{169}$. Therefore, $\sqrt{m}+\sqrt{n}=\sqrt{4}+\sqrt{169}=2+13=15$.

25. Let R be the right triangle with vertices at $(0,0)$, $(a,0)$, and $(0,b)$. The first integral can be interpreted as the volume obtained when R is revolved about the x-axis. The second integral can be interpreted as the volume obtained when R is revolved about the y-axis. Both solids of revolution product cones, where a and b play the part of radius or height, depending on the axis of rotation. In this case, we have $\frac{1}{3}\pi b^2a=800\pi$ and $\frac{1}{3}\pi a^2b=1920\pi$. Solving simultaneously gives $a=24$ and $b=10$, so $a^2+b^2=576+100=676$.

26. Since $y=x^3-x$, $y'=3x^2-1$, and therefore $y'(0)=-1$. Since we need the slope of the line to be negative and intersect the graph in three points, we must have $-1<m<0$. The graphs of $y=x^3-x$ and $y=mx$ intersect in the fourth quadrant when $x=0$ or $x=\sqrt{m+1}$. The area enclosed in the fourth quadrant is

$\int_0^1 \left(0-\left(x^3-x\right)\right)dx = \left(\frac{1}{2}x^2 - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{2}-\frac{1}{4}=\frac{1}{4}$, so the area enclosed only above or below the line is $\frac{1}{8}$. Therefore, $\frac{1}{8}=\int_0^{\sqrt{m+1}}\left(mx-\left(x^3-x\right)\right)dx = \left(\frac{(m+1)x^2}{2}-\frac{x^4}{4}\right)\Big|_0^{\sqrt{m+1}}$

$= \frac{(m+1)^2}{2}-\frac{(m+1)^2}{4}=\frac{(m+1)^2}{4} \Rightarrow (m+1)^2=\frac{1}{2} \Rightarrow m+1=\frac{\sqrt{2}}{2} \Rightarrow m=-1+\frac{\sqrt{2}}{2}=\frac{\sqrt{2}-2}{2}$.

27. Let $P\left(a,a^2\right)$ be the vertex of the triangle in the first quadrant. The vertex in the second quadrant must be $\left(-a,a^2\right)$ in order to have an equilateral triangle, so the side length of the triangle is $2a$, and the third vertex is the y-intercept of either line. Considering the line with positive slope, the slope of this line must be $\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$, but the slope must also be $2a$ because it is a tangent. Therefore, $2a=\sqrt{3}$, and the area enclosed by the triangle is $\frac{\left(\sqrt{3}\right)^2\sqrt{3}}{4}=\frac{3\sqrt{3}}{4}$.

28. By the washer method, the volume is $V=\pi\int_1^\infty\left(\left(\frac{1}{g(x)}\right)^2-\frac{1}{x^2}\right)dx$, and $\int_1^\infty \frac{1}{x^2}dx$ $=\lim_{t\to\infty}\left(-\frac{1}{x}\right)\Big|_1^t = \lim_{t\to\infty}\left(1-\frac{1}{t}\right)=1$. Graphing $y=\left(g(x)\right)^{-1}$, we see that $\int_1^\infty\left(\frac{1}{g(x)}\right)^2 dx$ $=\lim_{t\to\infty}\int_1^t\left(\frac{1}{g(x)}\right)^2 dx = \lim_{t\to\infty}\sum_{n=1}^{t}\frac{1}{n^2}=\frac{\pi^2}{6}$, so the volume is $\pi\left(\frac{\pi^2}{6}-1\right)=\frac{\pi^3}{6}-\pi$.

29. Referencing what was done in problem 11, the parabola could be the one with vertex $(1,1)$ and x-intercepts $(\pm 1,0) \Rightarrow y=1-x^2$. Therefore the area of the rectangle is $A=2xy=2x\left(1-x^2\right)=2x-2x^3 \Rightarrow A'=2-6x^2$. A sign chart shows that, in context, the maximum occurs when $x=\frac{\sqrt{3}}{3}$, so the area is $A=2\left(\frac{\sqrt{3}}{3}\right)\left(1-\frac{1}{3}\right)=\frac{4\sqrt{3}}{9}$.

30. In order for the graphs to not intersect, we must have imaginary values for their intersection. Setting the two equal, $x^2+(2b-2a)x+(1-2ab)=0$, so the discriminant must satisfy $(2b-2a)^2-4(1)(1-2ab)<0 \Rightarrow 4a^2+4b^2-4<0 \Rightarrow$

$a^2+b^2<1$. Since a and b were nonnegative, this area is $\frac{1}{4}\pi(1)^2=\frac{\pi}{4}$. Since $b\le 4-a^2$, and again that a and b were nonnegative, this area is selected out of a total area of $\int_0^2(4-a^2)\,da=\left(4a-\frac{1}{3}a^3\right)\Big|_0^2=8-\frac{8}{3}=\frac{16}{3}$. Therefore, the probability is $\frac{\pi/4}{16/3}=\frac{3\pi}{64}$.