Answers:

1. B
2. C
3. A
4. A
5. D
6. B
7. C
8. D
9. A
10. E
11. C
12. B
13. C
14. D
15. D
16. D
17. E
18. A
19. D
20. B
21. E
22. B
23. D
24. A
25. C
26. E
27. A
28. B
29. D
30. B

Solutions:

1. $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}+\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}=\begin{bmatrix}1+5 & 2+6\\ 3+7 & 4+8\end{bmatrix}=\begin{bmatrix}6 & 8\\ 10 & 12\end{bmatrix}$

2. $\begin{bmatrix}2 & 3 & 4\\ 1 & 3 & 6\\ 3 & 4 & 2\end{bmatrix}\cdot\begin{bmatrix}1 & 4 & 3\\ 3 & 2 & 4\\ 1 & 5 & 7\end{bmatrix}=\begin{bmatrix}2+9+4 & 8+6+20 & 6+12+28\\ 1+9+6 & 4+6+30 & 3+12+42\\ 3+12+2 & 12+8+10 & 9+16+14\end{bmatrix}=\begin{bmatrix}15 & 34 & 46\\ 16 & 40 & 57\\ 17 & 30 & 39\end{bmatrix}$

3. $0=x^3-2x^2-9x+18=(x-3)(x+3)(x-2)\Rightarrow x=2,\ 3,\text{ or }-3$. $\begin{vmatrix}1 & 2 & 3\\ 4 & x & 6\\ 7 & 8 & 9\end{vmatrix}=9x+84$ $+96-21x-48-72=-12x+60$, which is largest when $x=-3$, and the determinant is $-12(-3)+60=96$.

4. $A^{-1}=\dfrac{1}{9-10}\begin{bmatrix}9 & -2\\ -5 & 1\end{bmatrix}=\begin{bmatrix}-9 & 2\\ 5 & -1\end{bmatrix}$

5. Since A is 3×3, $|3A|=3^3|A|=27\cdot8=216$

6. The trace is the sum of the entries on the main diagonal, so it is $1+2+3+4=10$.

7. $F_{1,3}$ is the entry in the first row, third column, so it is 10.

8. F^T is the transpose of F, so $F^T=\begin{bmatrix}1 & 3 & 0\\ 2 & 0 & 1\\ 10 & 2 & 5\end{bmatrix}$.

9. $$F^{-1}=\frac{1}{30-2-30}\begin{bmatrix}\begin{vmatrix}0 & 1\\ 2 & 5\end{vmatrix} & -\begin{vmatrix}2 & 1\\ 10 & 5\end{vmatrix} & \begin{vmatrix}2 & 0\\ 10 & 2\end{vmatrix}\\ -\begin{vmatrix}3 & 0\\ 2 & 5\end{vmatrix} & \begin{vmatrix}1 & 0\\ 10 & 5\end{vmatrix} & -\begin{vmatrix}1 & 3\\ 10 & 2\end{vmatrix}\\ \begin{vmatrix}3 & 0\\ 0 & 1\end{vmatrix} & -\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix} & \begin{vmatrix}1 & 3\\ 2 & 0\end{vmatrix}\end{bmatrix}=-\frac{1}{2}\begin{bmatrix}-2 & 0 & 4\\ -15 & 5 & 28\\ 3 & -1 & -6\end{bmatrix}$$

$$=\begin{bmatrix}1 & 0 & -2\\ 7.5 & -2.5 & -14\\ -1.5 & 0.5 & 3\end{bmatrix}$$

10. $0=\begin{vmatrix} 2 & 4 & 9 \\ 6 & 3 & x \\ 1 & 1 & 5 \end{vmatrix}=30+4x+54-27-2x-120=2x-63 \Rightarrow x=31.5$

11. $$\text{adj.}(A)=\begin{bmatrix} \begin{vmatrix} 1 & 4 \\ 7 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 4 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 1 \\ 2 & 7 \end{vmatrix} \\ -\begin{vmatrix} 2 & 9 \\ 7 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 9 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 2 & 7 \end{vmatrix} \\ \begin{vmatrix} 2 & 9 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 9 \\ 4 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 4 & 1 \end{vmatrix} \end{bmatrix}=\begin{bmatrix} -26 & 0 & 26 \\ 59 & -12 & -17 \\ -1 & 24 & -5 \end{bmatrix}$$

12. Because this is an upper-triangular matrix, the determinant is the product of the entries on the main diagonal. $1\cdot 2\cdot 3\cdot 4\cdot 5=120$

13. Idempotent means that the square of the matrix will equal the matrix. Therefore, $\begin{bmatrix} -1 & x \\ -1 & x \end{bmatrix}=\begin{bmatrix} -1 & x \\ -1 & x \end{bmatrix}^2=\begin{bmatrix} 1-x & x^2-x \\ 1-x & x^2-x \end{bmatrix} \Rightarrow x=2$.

14. Eigenvalues are the values λ such that $0=|A-\lambda I|$, where I is the appropriate identity matrix. Therefore, $0=\begin{vmatrix} 3-\lambda & 1 \\ 3 & 5-\lambda \end{vmatrix}=15-8\lambda+\lambda^2-3=\lambda^2-8\lambda+12$ $=(\lambda-6)(\lambda-2) \Rightarrow \lambda=2$ or $\lambda=6$.

15. $$(AB)^{-1}=B^{-1}A^{-1}=\begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & 2 \\ 7 & 8 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 4 & 3 & 9 \\ 7 & 8 & 2 \end{bmatrix}=\begin{bmatrix} 2+21 & 4+24 & 10+6 \\ 4+4+14 & 8+3+16 & 20+9+4 \\ 7+32 & 14+24 & 35+72 \end{bmatrix}$$
$$\begin{bmatrix} 23 & 28 & 16 \\ 22 & 27 & 33 \\ 39 & 38 & 107 \end{bmatrix}$$

16. The only product whose matrices' inner dimension are equal is CA.

17. The matrix is not square, and thus it has no determinant.

18. $$(AB)^T=B^TA^T=\begin{bmatrix} 1 & 3 & 5 \\ 7 & 8 & 2 \\ 4 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 1 \\ 7 & 8 & 5 \end{bmatrix}=\begin{bmatrix} 1+9+35 & 2+12+40 & 3+25 \\ 7+24+14 & 14+32+16 & 8+10 \\ 4+7 & 8+8 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 45 & 54 & 28 \\ 45 & 62 & 18 \\ 11 & 16 & 5 \end{bmatrix}$$

19. Using Cramer's Rule, $w = \dfrac{\begin{vmatrix} 4 & 1 & 3 \\ 3 & 2 & 4 \\ 6 & 6 & 1 \end{vmatrix}}{\begin{vmatrix} 4 & 1 & 3 \\ 3 & 2 & 4 \\ 6 & 4 & 1 \end{vmatrix}} = \dfrac{8+24+54-36-96-3}{8+24+36-36-64-3} = \dfrac{-49}{-35} = \dfrac{7}{5}$

20. The resulting determinant is created by switching the first and last rows, thus negating the determinant. Therefore, the answer is -36.

21. Symmetric matrices satisfy $A = A^T$, so $x = a_{3,2} = a_{2,3} = 0$.

22. Square matrices and symmetric matrices can have any entries on their diagonals, so the trace does not necessarily equal 0. The matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ is a singular matrix because its determinant is 0, but its trace is 15. A skew-symmetric matrix satisfies $A = -A^T$, but since transposing doesn't change the entries on the main diagonal, every entry must be its own negative, which is only possible if all the main diagonal entries are 0, which means the trace must be 0 also.

23. The matrix is square (IV) and singular (I – it has two identical rows). It is not idempotent because $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}^2 = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \neq \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. It is also not skew-symmetric because of what was found in question 22. Therefore, I & IV are the only accurate descriptions of the matrix.

24. Since $|A| = 36 - 10 - 24 = 2$, $|A^{-1}| = 1/2$.

25. $1+2+4+9+8+7+3+0+3+4+6+9+1+4+7+3=71$

26. The matrix is not skew-symmetric (I) since its main diagonal entries are non-zero. It is not singular (II) since it has determinant $35+72-27-60=20 \neq 0$. It is not

symmetric (III) because $a_{1,2}=2\neq 6=a_{2,1}$. The matrix is square, though, because it has 3 rows and 3 columns. Therefore, I, II, & III do not describe the matrix.

27. The new determinant replaced the old row 3 with the sum of the old rows 1 and 3, which does not change the determinant. It is still 10.

28. $0=\begin{vmatrix} x & 4 & 6 \\ 1 & x & 3 \\ 0 & 7 & 10 \end{vmatrix}=10x^2+42-21x-40=10x^2-21x+2=(10x-1)(x-2)\Rightarrow x=\frac{1}{10}$ or $x=2$. Since the question asked for the number of positive integers, it is 1.

29. $\begin{vmatrix} 1 & 7 \\ 9 & 8 \end{vmatrix}=8-63=-55$

30. Since transposing and finding the inverse of products reverses the corresponding orders of the matrices, and since there are an even number of these operations, just apply the transposition and the inverse operations to each individual matrix. For a 2×2 invertible matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $\left(\left(\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T\right)^{-1}\right)^T\right)^{-1}=\left(\left(\begin{bmatrix} a & c \\ b & d \end{bmatrix}^{-1}\right)^T\right)^{-1}$

$$=\left(\left(\frac{1}{ad-bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}\right)^T\right)^{-1}=\left(\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)^{-1}=\frac{1}{\left(\frac{1}{ad-bc}\right)}\cdot\frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so applying the transpose, then the inverse a total of four times has no effect on each matrix. Therefore, $\left(\left(\left(\left(\left(\left(\left((ABC)^T\right)^{-1}\right)^T\right)^{-1}\right)^T\right)^{-1}\right)^T\right)^{-1}=ABC$

$$=\begin{bmatrix} -3 & -2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}=\begin{bmatrix} -3-6 & -6-8 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}=\begin{bmatrix} -9 & -14 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$=\begin{bmatrix} -45-98 & -54-112 \\ -5-14 & -6-16 \end{bmatrix}=\begin{bmatrix} -143 & -166 \\ -19 & -22 \end{bmatrix}.$$