Answers:

1. B
2. D
3. A
4. C
5. A
6. B
7. C
8. D
9. D
10. C
11. B
12. D
13. C
14. D
15. D
16. A
17. E
18. A
19. C
20. D
21. B
22. B
23. C
24. C
25. D
26. A
27. A
28. C
29. C
30. D

Solutions:

1. $\left(97+87+77+67\right)-\left(7+17+27+37\right)=\left(97-37\right)+\left(87-27\right)+\left(77-17\right)+\left(67-7\right)$ $=4\cdot 60=240$

2. The common difference of the sequence is 3, so $2011=-8+3\left(n-1\right)\Rightarrow 3\left(n-1\right)$ $=2019\Rightarrow n-1=673\Rightarrow n=674$.

3. The 10th term would be $6\left(12\right)^{9}=2\cdot 3\cdot 2^{18}\cdot 3^{9}=2^{19}\cdot 3^{10}$.

4. $1=2011+2010d\Rightarrow 2010d=-2010\Rightarrow d=-1$, so $a_{100}=2011+99\left(-1\right)=2011-99$ $=1912$.

5. The two common ratios would be negatives of each other, so whatever the value of the common ratio, the two possible first terms would be negative of each other, causing the sum to be 0.

6. Performing the computation in base 3, $t_2=10_3\cdot 2_3+2_3=22_3$, $t_3=10_3\cdot 22_3+2_3=222_3$, $t_4=10_3\cdot 222_3+2_3=2222_3$, $t_5=10_3\cdot 2222_3+2_3=22222_3$, and $t_6=10_3\cdot 22222_3+2_3$ $=222222_3$.

7. $15+16+17+...+n=15n\Rightarrow \dfrac{n\left(n+1\right)-14\left(15\right)}{2}=15n\Rightarrow n^2+n-210=30n\Rightarrow 0=$ $n^2-29n-210=\left(n-35\right)\left(n+6\right)\Rightarrow n=35$ since n was a positive integer.

8. The smallest multiple is $9\cdot 12=108$ and the largest is $9\cdot 111=999$, so the sum of all the multiples of 9 is $9\cdot\dfrac{111-12+1}{2}\left(12+111\right)=9\cdot 50\cdot 123=55{,}350$.

9. $\sum_{n=7}^{17}\left(7n+17\right)=\dfrac{17-7+1}{2}\left(66+136\right)=5.5\cdot 202=1111$

10. $k=39^2-4=\left(39-2\right)\left(39+2\right)=37\cdot 41$, and the smallest repunit that is a multiple of each of these factors are $111=37\cdot 3$ and $11111=41\cdot 271$. Since these two repunits have 3 and 5 1's, and since the greatest common divisor of 3 and 5 is 1, the smallest repunit that is a multiple of both has $3\cdot 5=15$ 1's.

11. It appears that the sequence is made up of the Fibonacci numbers, and this pattern continues since $\binom{6}{0}+\binom{5}{1}+\binom{4}{2}+\binom{3}{3}=1+5+6+1=13$.

12. If the three numbers are written as $a-d$, a, and $a+d$, then $21=a-d+a+a+d=3a$ $\Rightarrow a=7$. Now, $2=\dfrac{(a+d)a}{a(a-d)}\Rightarrow 2a-2d=a+d\Rightarrow 7=a=3d\Rightarrow d=\dfrac{7}{3}$, so the largest of the three numbers is $a+d=7+\dfrac{7}{3}=\dfrac{28}{3}$.

13. The coefficients form the start of the Tribonacci sequence, so adding the three equations together yields $4a+7b+13c=24$, then adding that to the second two given equations yields $7a+13b+24c=44$.

14. $\dfrac{2(10\cdot 1+9\cdot 2+8\cdot 3+7\cdot 4+6\cdot 5)}{1+2+3+4+5+6+7+8+9+10}=\dfrac{2(10+18+24+28+30)}{\frac{10}{2}(1+10)}=\dfrac{220}{55}=4$

15. If the side length of the first square is labeled s, then the area enclosed between the first shapes is $s^2-\pi\left(s/2\right)^2=s^2\left(1-\dfrac{\pi}{4}\right)$. For each square and circle after that, the side and radius are at a scale factor of $\dfrac{\sqrt{2}}{2}$ from the original, so the corresponding area factor is $\left(\sqrt{2}/2\right)^2=\dfrac{1}{2}$, so the areas form an infinite geometric sequence whose sum is $\dfrac{s^2\left(1-\frac{\pi}{4}\right)}{1-\frac{1}{2}}=s^2\left(2-\dfrac{\pi}{2}\right)$. The proportion of shaded area from the original square interior then would be $\dfrac{s^2\left(2-\frac{\pi}{2}\right)}{s^2}=2-\dfrac{\pi}{2}$.

16. $-300=\sum_{k=0}^{2}(a+kd)=3a+3d$ and $300=\sum_{k=0}^{8}(a+kd)=9a+36d$. Multiplying the first equation by 3 and subtracting from the second equation yields $1200=27d$ $\Rightarrow 400=9d$. Therefore, $\sum_{k=0}^{5}(a+kd)=6a+15d=(6a+6d)+9d=2\cdot -300+400$ $=-600+400=-200$.

17. $1=b_1+b_2=b_1+b_1r=b_1(1+r)\Rightarrow b_1=\frac{1}{1+r}$, so $2=\sum_{k=1}^{\infty}b_k=\frac{1}{(1+r)(1-r)}=\frac{1}{1-r^2}$

$1-r^2=\frac{1}{2}\Rightarrow r^2=\frac{1}{2}\Rightarrow r=\pm\frac{\sqrt{2}}{2}$. If $b_1<0$, then the sum would be negative, so

$b_1>0\Rightarrow r<0\Rightarrow b_1=\frac{1}{1+r}=\frac{1}{1-\sqrt{2}/2}=\frac{2}{2-\sqrt{2}}=2+\sqrt{2}$.

18. Using the method of finite differences,

$p(-2)$	1	15	133	427
$1-p(-2)$	14	118	294	
$13+p(-2)$	104	176		
$91-p(-2)$	72			

Because it is known that p is cubic, this last row must consist of the same terms. Therefore, $91-p(-2)=72\Rightarrow p(-2)=19$.

19. $c_{10}=\sum_{m=1}^{10}b_m=\sum_{m=1}^{10}\sum_{n=1}^{m}a_n=\sum_{m=1}^{10}\sum_{n=1}^{m}2^k=\sum_{m=1}^{10}\left(2^{m+1}-2\right)=4\left(2^{10}-1\right)-10\cdot 2=4096-4-20$
$=4072$

20. $a>10$ and $ar^4<1000\Rightarrow 10r^4<ar^4<1000\Rightarrow r^4<100$, so the integral values of r that could work are 0, ±1, ±2, and ±3. Making $a=11$, we can see that both inequalities would be satisfied from all 7 values of r.

21. The sequence of remainders when the Fibonacci numbers are divided by 8 is 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, 0, 1, 1, …, and the two 1's start the repeat of the initial sequence, so the only remainders possible are 0, 1, 2, 3, 5, 7, and $0+1+2+3+5+7=18$.

22. The sequence of remainders when the sum of the digits of these terms are divided by 3 is 1, 1, 0, 1, 1, 0, …, so every 3rd number is divisible 3. Therefore, the 100th term divisible by 3 would be the 300th term.

23. $S=\sum_{k=1}^{8}\log_5 k=\log_5 8!=\log_5 40320$. Since $5^6=15625$ and $5^7=78125$, we must have $6<S<7$. Now, $S-6=\log_5\frac{40320}{15625}>\log_5\frac{40000}{15625}=\log_5\frac{64}{25}=\log_5 2.56$, and since $2.56^2>5$, we must have $\log_5 2.56^2>1\Rightarrow\log_5 2.56>1/2\Rightarrow S-6>1/2\Rightarrow S>6.5$, so the closest integer value to S is 7.

24. $\left\lfloor \dfrac{2011!}{2010!+2009!+2008!+\ldots+1!} \right\rfloor = \left\lfloor \dfrac{2011}{1+\frac{1}{2010}+\frac{1}{2010\cdot 2009}+\ldots+\frac{1}{2010!}} \right\rfloor$, and let D be this new denominator. Clearly, $2010=\dfrac{2011}{1+\frac{1}{2010}}$ $>\dfrac{2011}{1+\frac{1}{2010}+\frac{1}{2010\cdot 2009}+\ldots+\frac{1}{2010!}}=\dfrac{2011}{D}$, so 2010 is an upper bound for the fraction. To get a lower bound for the fraction, we must compare to an infinite geometric sequence, and we eventually get the inequality $D<1+\dfrac{1}{2000}<1+\dfrac{1}{2000}+\dfrac{1}{2000^2}+\ldots=\dfrac{1}{1-\frac{1}{2000}}=\dfrac{2000}{1999}$. The second inequality sign is obvious, and for the first inequality sign, note that if we started subtracting terms of D from $\dfrac{1}{2000}$, we could see that the numerators become large. For example, $\dfrac{1}{2000}-\dfrac{1}{2010}=\dfrac{10}{2000\cdot 2010}$, $\dfrac{1}{2000}-\dfrac{1}{2010}-\dfrac{1}{2010\cdot 2009}=\dfrac{10\cdot 2009-2000}{2000\cdot 2010\cdot 2009}$, etc. This leads to the conclusion that $2009<2011-\dfrac{2011}{2000}=\dfrac{2011}{2000/1999}<\dfrac{2011}{D}$, so 2009 is a lower bound for the fraction. The upper and lower bounds for the fraction together imply $\left\lfloor \dfrac{2011!}{2010!+2009!+2008!+\ldots+1!} \right\rfloor=2009$.

25. $a_{428}=S_{428}-S_{427}=4\left(428^2-427^2\right)+2\left(428-427\right)+\left(8-8\right)=4\cdot 855+2=3422$

26. As the ball falls, it falls down the steps, so the distance it travels downward is sometimes farther than the distance it traveled upward. Let a, b, c, and d be the four additional distances the ball travels downward as it falls down the steps, so $a+b+c+d=2\cdot 10=20\text{ feet}$. There are many ways this could happen, but in all cases the vertical distance traveled is the same. For instance, the distance is $32+\frac{1}{2}\cdot 32+\left(\frac{1}{2}\cdot 32+a\right)+\left(\frac{1}{4}\cdot 32+\frac{1}{2}a\right)+\left(\frac{1}{4}\cdot 32+\frac{1}{2}a+b\right)+\ldots$, which can be computed in pieces as :

$$32+\frac{1}{2}\cdot 32+\frac{1}{2}\cdot 32+\frac{1}{4}\cdot 32+\frac{1}{4}\cdot 32+\ldots=32\left(\frac{1}{1-1/2}+\frac{1/2}{1-1/2}\right)=32(3)=96$$

$$a+\frac{1}{2}\cdot a+\frac{1}{2}\cdot a+\frac{1}{4}\cdot a+\frac{1}{4}\cdot a+\ldots=a\left(\frac{1}{1-\frac{1}{2}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)=a(3)=3a$$

$$b+\frac{1}{2}\cdot b+\frac{1}{2}\cdot b+\frac{1}{4}\cdot b+\frac{1}{4}\cdot b+\ldots=b\left(\frac{1}{1-\frac{1}{2}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)=b(3)=3b$$

$$c+\frac{1}{2}\cdot c+\frac{1}{2}\cdot c+\frac{1}{4}\cdot c+\frac{1}{4}\cdot c+\ldots=c\left(\frac{1}{1-\frac{1}{2}}+\frac{\frac{1}{2}}{1-\frac{1}{2}}\right)=c(3)=3c$$

Therefore, the total distance covered is $96+3a+3b+3c=96+3(a+b+c)$ $=96+3\cdot 20=96+60=156$, which is the same distance the ball would have traveled if the steps weren't there and the ball feel freely from a height equal to $32+20$ feet.

27. The first term is $\frac{8}{r^2}$, so the sum is $\frac{8}{r^2(1-r)}$. Using the AM-GM inequality, we know that $\frac{r+r+(2-2r)}{3}\geq\sqrt[3]{r^2(2-2r)}\Rightarrow\frac{2}{3}\geq\sqrt[3]{2r^2(1-r)}\Rightarrow\frac{8}{27}\geq 2r^2(1-r)\Rightarrow\frac{4}{27}\geq r^2(1-r)$ and equality holds if $r=2-2r\Rightarrow 3r=2\Rightarrow r=\frac{2}{3}$, which does not contradict convergenece. Therefore, the largest sum is $\frac{8}{\left(\frac{2}{3}\right)^2\left(1-\frac{2}{3}\right)}=\frac{8}{\frac{4}{27}}=54$.

28. We know that $2a-b=c-2a\Rightarrow 4a=b+c$, and we also know that $a^2+b^2=c^2$, which implies $a^2=c^2-b^2=(c+b)(c-b)=4a(c-b)\Rightarrow c-b=\frac{a}{4}$. Adding the first equation to this equation yields $2c=4a+\frac{a}{4}=\frac{17}{4}a\Rightarrow c=\frac{17}{8}a\Rightarrow b=\frac{15}{8}a$, so the Pythagorean triples are of the form $(8d,15d,17d)$. Since the length of the hypotenuse must be less than 100, this would occur if a is 1, 2, 3, 4, or 5, so there are 5 possibilities.

29. After each four moves the lemur makes a ring of a spiral. The coordinates of the $4k$th node is $(1-3+5-\ldots-(4k-1),2-4+6-\ldots-4k)=(-2k,-2k)$. Similarly, the $(4k+1)$st node is $(2k+1,-2k)$, the $(4k+2)$nd node is $(2k+1,2k+2)$, and the $(4k+3)$rd node is $(-2k-2,2k+2)$. Disregarding signs and the ordering of each pair, the goal is to determine when points of the form (m,m) or $(n,n+1)$ are an integer distance from the origin. No points of the first form are an integer distance away since they would be a distance of $m\sqrt{2}$ away, which is never an integer. For the

second form, we must find Pythagorean triples whose leg lengths are consecutive integers. The first few of these are $(3,4,5)$, $(20,21,29)$, and $(119,120,169)$, but the last one of these is beyond the last node. Therefore, including the point $(1,0)$, there are a total of 3 nodes whose distance from the origin is an integer length.

30. Since $\sin 180^\circ$ and $\sin x = \sin(180^\circ - x)$, we have $2\sin 2^\circ + 4\sin 4^\circ + 6\sin 6^\circ + \ldots + 180\sin 180^\circ = (2\sin 2^\circ + 88\sin 178^\circ) + (4\sin 4^\circ + 86\sin 176^\circ) + \ldots + (88\sin 88^\circ + 2\sin 92^\circ) + 90\sin 90^\circ + 90\sin 92^\circ + \ldots + 90\sin 178^\circ = 90\sin 2^\circ + 90\sin 4^\circ + \ldots + 90\sin 178^\circ$, which we will call S. Multiplying both sides of this by $\sin 1^\circ$, we get $S\sin 1^\circ = 90\sin 2^\circ \sin 1^\circ + \ldots 90\sin 178^\circ \sin 1^\circ$, and using the product-to-sum formula $2\sin x \sin 1^\circ = \cos(x - 1^\circ) - \cos(x + 1^\circ)$, we get
$S\sin 1^\circ = 45(2\sin 2^\circ \sin 1^\circ + \ldots + 2\sin 178^\circ \sin 1^\circ) = 45((\cos 1^\circ - \cos 3^\circ) + (\cos 3^\circ - \cos 5^\circ) + \ldots + (\cos 177^\circ - \cos 179^\circ)) = 45(\cos 1^\circ - \cos 179^\circ) = 45(\cos 1^\circ + \cos 1^\circ) = 90\cos 1^\circ$.

Therefore, $S = \dfrac{90\cos 1^\circ}{\sin 1^\circ} = 90\cot 1^\circ$.