Answers:

1. B
2. D
3. B
4. C
5. B
6. A
7. B
8. C
9. A
10. D
11. D
12. C
13. C
14. C
15. D
16. B
17. A
18. C
19. C
20. C
21. A
22. D
23. B
24. B
25. A
26. B
27. C
28. A
29. B
30. A

Solutions:

1. There are a total of $\frac{8(8-3)}{2}=20$ diagonals in the octagon, and the only ones that pass through the center would be the ones whose vertices are diametrically opposite. There are 4 such pairs, so there are a total of $20-4=16$ diagonals that do not pass through the center of the circle.

2. By the Fundamental Counting Principle, there are $8\cdot 6\cdot 10=480$ such ways.

3. Similarly, there are $\binom{8}{2}\cdot 6\cdot 10=28\cdot 6\cdot 10=1680$ such ways since wearing the same two shirts, regardless of which one is on the outside, is considered the same outfit.

4. Let R, B, and K be the sets of birds with red, blue, or black feathers, respectively. Then $|R\cup B\cup K|=|R|+|B|+|K|-|R\cap B|-|R\cap K|-|B\cap K|+|R\cap B\cap K|=60+60+75$ $-15-25-20+5=140$ total birds.

5. $\frac{4}{6}\cdot\frac{2}{5}=\frac{8}{30}=\frac{4}{15}$

6. $12^3=1728$ and $13^3=2197$, so 13^3 is the first to be counted. $39^3=59319$ and $40^3=64000$, so 39^3 is the last to be counted. Therefore, there are a total of $39-13+1=27$ perfect cubes that fit the criterion.

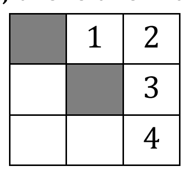
7. Since every factor of 80 consists of four 2s and a 5, we must find how many of each prime are in the prime factorization of 2011!. By Legendre's theorem, the number of 5s is $\left\lfloor\frac{2011}{5}\right\rfloor+\left\lfloor\frac{2011}{25}\right\rfloor+\left\lfloor\frac{2011}{125}\right\rfloor+\left\lfloor\frac{2011}{625}\right\rfloor=402+80+16+3=501$, and the number of 2s is $\left\lfloor\frac{2011}{2}\right\rfloor+\left\lfloor\frac{2011}{4}\right\rfloor+\left\lfloor\frac{2011}{8}\right\rfloor+\left\lfloor\frac{2011}{16}\right\rfloor+\left\lfloor\frac{2011}{32}\right\rfloor+\left\lfloor\frac{2011}{64}\right\rfloor+\left\lfloor\frac{2011}{128}\right\rfloor$ $+\left\lfloor\frac{2011}{256}\right\rfloor+\left\lfloor\frac{2011}{512}\right\rfloor+\left\lfloor\frac{2011}{1024}\right\rfloor=1005+502+251+125+62+31+15+7+3+1=2002$. If we use 500 of the 5s and 2000 of the 2s, this would be 500 80s, with one 5 and two 2s left over. Therefore, the largest number of 80s in the factorization is 500.

8. There are a total of $\frac{20\cdot 3}{2}=30$ edges since each edge is used for two faces. Now, since 5 edges radiate from each vertex, and each edge radiates from two vertices,

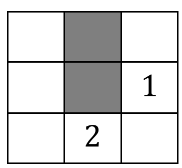
there are a total of $\frac{30 \cdot 2}{5} = 12$ vertices.

9. There are $\binom{5}{2} = 10$ subsets that have 1 as their smallest element, $\binom{4}{2} = 6$ subsets that have 2 as their smallest element, $\binom{3}{2} = 3$ subsets that have 3 as their smallest element, and $\binom{2}{2} = 1$ subset that has 4 as its smallest element. Therefore,
$\sum_{k=1}^{20} \min(A_k) = 10 \cdot 1 + 6 \cdot 2 + 3 \cdot 3 + 1 \cdot 4 = 10 + 12 + 9 + 4 = 35.$

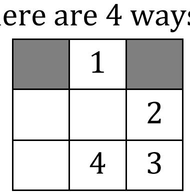
10. 1) If the center square is to be shaded, our goal is to arrange two shaded squares on the outside of the center square. This case is easier if we rotate any position so that the greatest possible number of squares on the top row are shaded. If a corner square on the top row is shaded, there are 4 distinct ways to shade another square:



If the figure cannot be rotated such that a corner square is shaded on the top row, then only side squares are shaded. Rotate one of them to the top row and count 2 distinct arrangements:

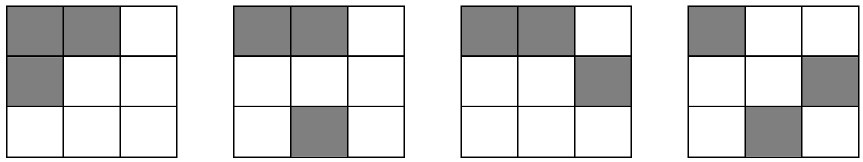


2) Now, if the center square is not to be shaded, we will begin by rotating as many squares as possible to the top row to make the counting easier. If both of the top two corners are shaded, then there are 4 ways to shade a third square:



If one top corner square is shaded, along with the opposite corner and no other corners, then there is only 1 arrangement because all others are indistinct.

If only one corner square is shaded, then there are four arrangements. We could break these into subsubcases if necessary (at least one shaded side square is adjacent to the shaded corner square vs. the corner square and two nonadjacent side squares are shaded):



Finally, if no corners are shaded, then three side squares are shaded. All such arrangements are indistinct, so there is only 1 arrangement.

Adding these cases together, we get a total of $4+2+4+1+4+1=16$ distinct arrangements.

11. We need only set the first and second digits to determine the four-digit number, so let (a,b) be the ordered pair consisting of the first and second digits of the number. If $a=1$, we have 3 possibilities for b (1, 2, 3). If $a=2$, we have 3 possibilities for b (2, 3, 4). If $a=3$, we have 4 possibilities for b (2, 3, 4, 5). If $a=4$, we have 3 possibilities for b (3, 4, 5). If $a=5$, we have 3 possibilities for b (4, 5, 6). If $a=6$, we have 4 possibilities for b (4, 5, 6, 7). If $a=7$, we have 3 possibilities for b (5, 6, 7). If $a=8$, we have 3 possibilities for b (6, 7, 8). If $a=9$, we have 4 possibilities for b (6, 7, 8, 9). Altogether, there are $3+3+4+3+3+4+3+3+4=30$ possibilities.

12. There are $8^3=512$ total possible rolls, and the only ways to roll a sum of 22 are with two 7s and an 8 or two 8s and a 6. In each of these cases there are $\binom{3}{1}=3$ positions to place the single number, so there are a total of 6 ways to roll a sum of 22, making the probability $6/512 = 3/256$.

13. The prime factorizations of 500 and 1200 are $5^3 2^2$ and $5^2 3^1 2^4$, respectively. Therefore, any positive integral divisors of 500 that are also divisors of 1200 must come from $5^2 2^2$ since that is the part of the factorization of 500 that is shared by 1200. Therefore, there are a total of $(2+1)(2+1)=3^2=9$ such divisors.

14. Since $\binom{25}{12}=\frac{25!}{12!13!}$, we can apply Legendre's theorem in order to count the primes in the prime factorization by adding the number of times the prime appears in the numerator and subtracting the number of times the prime appears in the denominator. For instance, the number of 2s in the numerator is $\left\lfloor \frac{25}{2} \right\rfloor + \left\lfloor \frac{25}{4} \right\rfloor$

$+\left\lfloor \frac{25}{8} \right\rfloor + \left\lfloor \frac{25}{16} \right\rfloor = 12+6+3+1=22$. For the denominator, the number of 2s is $\left\lfloor \frac{12}{2} \right\rfloor + \left\lfloor \frac{12}{4} \right\rfloor + \left\lfloor \frac{12}{8} \right\rfloor + \left\lfloor \frac{13}{2} \right\rfloor + \left\lfloor \frac{13}{4} \right\rfloor + \left\lfloor \frac{13}{8} \right\rfloor = 6+3+1+6+3+1=20$. Therefore, there are $22-20=2$ 2s in the prime factorization. Doing similar calculators for the other primes up through 19, there are $10-10=0$ powers of 3, $6-4=2$ powers of 5, $3-2=1$ power of 7, $2-2=0$ powers of 11, $1-1=0$ powers of 13, $1-0=1$ power of 17, and $1-0=1$ power of 19, making a total of 5 primes that are divisors of $\binom{25}{12}$ (those numbers are 2, 5, 7, 17, and 19).

15. Since the sum of all faces on both dice is $2(1+2+...+20)=420$, which is a multiple of 3, find the probability that the sum of the 38 showing faces is a multiple of 3 is the same as showing that the sum of the unseen faces is also 3. In other words, we are merely looking for the probability that the sum of faces rolled is a multiple of 3. Considering all the possible sums that are multiple of 3, there are 2 ways each of rolling a sum of 3 or 39, 5 ways each of rolling a sum of 6 or 36, 8 ways each of rolling a sum of 9 or 33, 11 ways each of rolling a sum of 12 or 30, 14 ways each of rolling a 15 or a 27, 17 ways each of rolling a sum of 18 or 24, and 20 ways of rolling a sum of 21. Therefore, the total number of ways of rolling a sum that is a multiple of 3 is $2(2+5+8+11+14+17)+20=134$, and the total number of different rolls is $20^2=400$, so the probability is $134/400 = 67/200$.

16. Noah must be included (1 way), and there are three ways to either include or not include Richard Dawkins and Jerry Falwell (everything except both on the raft). For the other 5 friends, there are 2 ways each to include or not include them, so the total number of ways to have people escape on the raft is $1\cdot 3\cdot 2^5=96$.

17. $\binom{7}{4}+\binom{7}{5}+\binom{7}{6}+\binom{7}{7}=35+21+7+1=64$

18. Let $S=\binom{15}{0}+3\binom{15}{1}+5\binom{15}{2}+...+(2n+1)\binom{15}{n}+...+31\binom{15}{15}$. Since $\binom{15}{n}=\binom{15}{15-n}$, $S=31\binom{15}{0}+29\binom{15}{1}+27\binom{15}{2}+...+\binom{15}{15}$ also. Adding these two equations together yields $2S=32\binom{15}{0}+32\binom{15}{1}+32\binom{15}{2}+...+32\binom{15}{15}=32\left(2^{15}\right)=2^{20} \Rightarrow S=2^{19}$.

19. To be a multiple of 9, the sum of the digits must be divisible by 9. Additionally, since the sum of all the possible digits is $0+1+2+3+4+5+6+7+8+9=45$, the digits we leave out of the number must also sum to a multiple of 9. Listing the digits in increasing order, there are 24 such combinations of numbers we could leave out: 10 that contain 0 (0126, 0135, 0234, 0189, 0279, 0369, 0378, 0459, 0468, 0567) and 14 that do not contain 0 (1269, 1278, 1359, 1368, 1458, 1467, 2349, 2358, 2367, 2457, 3456, 3789, 4689, 5679). Therefore, for the six-digit number, there are 10 that don't contain 0, each of which has $6!=720$ different arrangements, and 14 that do contain 0, each of which has $5\cdot 5!=600$ different arrangements. Therefore, the total number of numbers that fit the criteria are $10\cdot 720+14\cdot 600=15600$.

20. To find the greatest number of rectangles that could be formed, all of the lines must be parallel or perpendicular to one another. Suppose n lines are parallel in one direction. Then $15-n$ lines must be parallel in the other direction. Therefore, the number of rectangles would be $\binom{15-n}{2}\binom{n}{2}=\frac{(15-n)(14-n)}{2}\cdot\frac{n(n-1)}{2}$

$=\frac{(15-n)(n-1)}{2}\cdot\frac{(14-n)n}{2}=\frac{49-(n-8)^2}{2}\cdot\frac{49-(n-7)^2}{2}$, which is largest when either $n=7$ or $n=8$. Therefore, the largest number of rectangles would be $\binom{8}{2}\binom{7}{2}$ $=28\cdot 21=588$.

21. All terms in the expansion have terms of the form $x^a y^b$. The maximum possible value of b assumes that x^a is the result of as few terms as possible multiplied by y^2 terms. The minimum possible value of b assumes that x^a is the result of as many terms as possible multiplied by y^1 terms. For $0\le a<10$, the minimum possible value of b is $10-a$; for $10\le a\le 20$, the minimum possible value of b is 0 since all the terms multiplied could be x terms. In all cases, the maximum possible value of b results from $\left\lceil a/2\right\rceil$ x-terms multiplied together, so the number of remaining terms is $10-\left\lceil a/2\right\rceil$, resulting in a maximum value of b of $2\left(10-\left\lceil a/2\right\rceil\right)=20-2\left\lceil a/2\right\rceil$, which equals $20-a$ if a is even and $19-a$ if a is odd. The total number of terms is the sum of the maximal values of b minus the number of positive integers less than the minimal values of b for $0\le a\le 20$, which is $(20+18+18+16+16+\ldots+2+2+0)$ $-(9+8+7+\ldots+1+0+0+\ldots+0)=200-45=155$.

22. By the pigeonhole principle, there cannot be three circles of the same color. Therefore, the only possible cases are that 1) two circles are of one color, two circles

of another color, and the last circle of a third color; 2) two circles are of one color and the other three circles are all of three other different colors; and 3) all five circles are different colors.

In the first case, let circle 1 be the lone-colored circle. Circle 1 has 5 color options while the others must alternate colors. There are $5\cdot 4\cdot 3=60$ possibilities for colors, and since any of the five circles could be the lone-colored circle, there are a total of $5\cdot 60=300$ colorings in the first case.

In the second case, the two circles with the same color must have exactly 1 circle in between them, and there are 5 ways to select this pair. Additionally, there are $5\cdot 4\cdot 3\cdot 2=120$ ways of choosing the colors, so there are a total of $5\cdot 120=600$ colorings in the second case.

In the third case, there are simply $5!=120$ different colorings.

In total, there are $300+600+120=1020$ total colorings.

23. Let $n=p_1^{e_1}\cdot p_2^{e_2}\cdot \ldots \cdot p_k^{e_k}$, where $e_1\ge e_2\ge \ldots \ge e_k$. Since n has 4 positive divisors that are fifth powers, we have $\left\lfloor \frac{e_1}{5}+1\right\rfloor\cdot\left\lfloor \frac{e_2}{5}+1\right\rfloor\cdot\ldots\cdot\left\lfloor \frac{e_k}{5}+1\right\rfloor=4$. This product of positive integers equals 4, so no more than two of those integers is greater than 1. This means that no more than two of the exponents e_i are at least 5, which gives us two possible models for n: either 1) $15\le e_1<20$ and $e_i<5$ for $1<i\le k$, or 2) $5\le e_2\le e_1<10$ and $e_i<5$ for $2<i\le k$.

Next, since n has 6 positive divisors that are perfect cubes, we have $\left\lfloor \frac{e_1}{3}+1\right\rfloor\cdot\left\lfloor \frac{e_2}{3}+1\right\rfloor\cdot\ldots\cdot\left\lfloor \frac{e_k}{3}+1\right\rfloor=6$. Again, this product of positive integers is such that no more than two of those integers are greater than 1. According to our model, we have $\left\lfloor \frac{e_1}{3}+1\right\rfloor\cdot\left\lfloor \frac{e_2}{3}+1\right\rfloor=6$, and this product is either $6\cdot 1$ or $3\cdot 2$. Updating the two models, we have either 1) $15\le e_1<18$ and $e_i<3$ for $1<i\le k$, or 2) $6\le e_1<9$ and $e_2=5$.

Next, since n has 12 perfect square divisors, we have $\left\lfloor \frac{e_1}{2}+1\right\rfloor\cdot\left\lfloor \frac{e_2}{2}+1\right\rfloor\cdot\ldots\cdot\left\lfloor \frac{e_k}{2}+1\right\rfloor=12$. Attempting to fit this into the first model, since

$15 \le e_1 < 18,\ 8 \le \left\lfloor \frac{e_1}{2} + 1 \right\rfloor < 10$, but no integers in this range are divisors of 12, so this model isn't possible. Attempting to fit this into the second model, the only possibility is $\left\lfloor \frac{e_1}{2} + 1 \right\rfloor = 4$ and $\left\lfloor \frac{e_2}{2} + 1 \right\rfloor = 3$. Since we want the least possible number of positive divisors, $e_1 = 6$ and $e_2 = 5$, and $e_i = 0$ for all other i. Therefore, $n = p_1^6 \cdot p_2^5$, which has a total of $(6+1)(5+1) = 42$ positive divisors.

24. There are 5 possible rolls that have a sum of 8, only one of which is doubles (double 4s). Therefore, the probability is $1/5$.

25. It is easy to see that the equation of the plane is $x + y + z = 101$. Since each of x, y, and z must be positive, define $x' = x - 1,\ y' = y - 1,$ and $z' = z - 1$, then we are now looking for the number of ordered triples of nonnegative integers such that $x' + y' + z' = 98$. There are $\binom{98+3-1}{3-1} = \binom{100}{2}$ different ways to do this.

26. Just as in the last problem, define $x = a - 1,\ y = b - 1,$ and $z = c - 1$, which changes the equation to $x + y + 2z = 31$, and we are now looking for nonnegative integer solutions. Now, allowing z to be 0, 1, 2, …, 15 yields the equations $x + y = 31$, $x + y = 29$, ..., $x + y = 1$. The total number of solutions to these equations is $\binom{32}{1} + \binom{30}{1} + \binom{28}{1} + \ldots + \binom{2}{1} = 32 + 30 + 28 + \ldots + 2 = 272$.

27. There are a total of $\frac{7!}{2!2!3!} = 210$ to arrange the 7 beads in a row, but this is an overcounting since rotations will eliminate some arrangements. There is no rotational symmetry possible among 7 beads with multiple colors because 2 and 3 are not divisors of 7. So there are only $\frac{210}{7} = 30$ possible bracelets. Now some bracelets have symmetry when flipped. These are the ones that can be oriented with left-right symmetry such as BRGRGRB. A red bead must be in the middle. The first three beads must be matched symmetrically with the last three beads. There are $3! = 6$ ways to do this. So, flipping these bracelets does not result in a different ordering. However, each of the other $30 - 6 = 24$ patterns can be flipped to match one of the other 23 remaining patterns. This means that only $\frac{24}{2} = 12$ of the patterns are distinct. The total number of distinct bracelets, then, is $6 + 12 = 18$.

28. There are 12 players, each of whom watches 35 bouts. This means that a player watches a bout $12\cdot 35=420$ times. Since 10 players watch each bout, the number of bouts is $\frac{420}{10}=42$.

29. There are a total of $\binom{7}{3}=35$ ways of selecting the vertices. Now, each angle of the triangle is equal to half the number of degrees of the arc it intercepts. In the case of a heptagon, none of these arcs is equal to 180°, so none of the angles can be right. If the arc is greater than 180°, then the triangle is contained within a semicircle and does not contain the center of the circle. This means one of the angles is obtuse. One the other hand, every acute triangle formed by three of the vertices contains the center of the circle because any diameter drawn intersects two of the sides of the triangle.

At each vertex of the triangle formed by three vertices of the heptagon, we write a number equal to the number of vertices of the heptagon we must move clockwise until we encounter the next vertex of the triangle. The total must be 7. Since the triangle must be acute, none of these numbers can be greater than 3 (4 or more would imply an arc larger than 180°. These numbers must be permutations of the triples $(3,3,1)$ or $(3,2,2)$. The $(3,3,1)$ triangles are isosceles, and we can rotate one around the circle to form 7 possible acute triangles. The same is true for the $(3,2,2)$ triangles, so there are a total of $7+7=14$ acute triangles, making the probability $^{14}\!/_{35}={}^{2}\!/_{5}$.

30. The generating function for the roll of one of Nick's dice is $f(x)=x+x^2+x^3+x^4+x^5+x^6+x^7+x^8$, so the generating function for the sum is $f^2(x)=\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8\right)^2$. Let $p(x)$ and $g(x)$ be the generating functions for Bill's dice. Since the sums are the same, we known that $p(x)g(x)=\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8\right)^2$. Since 11 is the largest number on one of the dice, assume without loss of generality that the degree of $p(x)$ is 11. Now we are looking for an alternative factorization of the given polynomial. We have $p(x)g(x)=\left(x+x^2+x^3+x^4+x^5+x^6+x^7+x^8\right)^2=x^2\left(1+x+x^2+x^3+x^4+x^5+x^6+x^7\right)^2=x^2\left(\frac{x^8-1}{x-1}\right)^2=x^2\frac{\left(x^4+1\right)^2\left(x^4-1\right)^2}{(x-1)^2}$

$$= x^2 \frac{\left(x^4+1\right)^2\left(x^2+1\right)^2\left(x^2-1\right)^2}{\left(x-1\right)^2} = x^2 \frac{\left(x^4+1\right)^2\left(x^2+1\right)^2\left(x+1\right)^2\left(x-1\right)^2}{\left(x-1\right)^2}$$

$$= x^2\left(x^4+1\right)^2\left(x^2+1\right)^2\left(x+1\right)^2.$$

Now consider what we know about the polynomial functions of the individual dice. There are 8 sides, so the sum of the coefficients must be 8; this means that $p(1) = g(1) = 8$. Also, each coefficient must be nonnegative since you can't have a negative number of faces. Finally, neither polynomial can have a constant because the numbers on the faces must be natural numbers, and a constant has a degree of 0, which is not natural. This means that x must be a factor of each of $p(x)$ and $g(x)$.

The sums of the coefficients of the individual unsquared factors are 1, 2, 2, and 2 (as ordered in the last factorization). In order for $p(1) = g(1) = 8$, the factors with sums of 2 must be split evenly among $p(x)$ and $g(x)$. The sum of the degrees of the polynomials must be 16 and $p(x)$ has degree 11, so $g(x)$ has degree 5. The only way to get a degree of 5 with a factor of x and three factors with sum 2 is $g(x) = x(x+1)^2(x^2+1) = x + 2x^2 + 2x^3 + 2x^4 + x^5$, which makes $p(x) = x(x^2+1)(x^4+1)^2 = x + x^3 + 2x^5 + 2x^7 + x^9 + x^{11}$. Thus, one of Bill's dice has faces 1, 2, 2, 3, 3, 4, 4, 5; while the other has faces 1, 3, 5, 5, 7, 7, 9, 11. Therefore, the smaller sum is $1+2+2+3+3+4+4+5=24$.