Answers:

1. A
2. C
3. C
4. D
5. A
6. A
7. B
8. D
9. D
10. C
11. E
12. B
13. A
14. B
15. C

Solutions:

1. Complete a circle; i.e., prove A_1 implies A_2, A_2 implies A_3, …, A_{n-1} implies A_n, and A_n implies A_1. This consists of n proofs.

2. This proof consists of a basis case and an inductive step, which is a proof by induction.

3. There's nothing wrong with the first statement. Absurd though it may seem, it's a logically true statement. Nothing is wrong with the basis case either; given the method of proof, the basis case does hold true. The culprit is in the inductive step; notice that for $x=2$ and $y=1$, we get $\max(x-1, y-1)=\max(1,0)$. But we claim this property holds when both the arguments of $\max(\bullet,\bullet)$ are *positive* integers—this is why our basis case was $x=y=1$. So for $x=2$ and $y=1$, we can't conclude that $x-1=y-1$ since $y-1$ is not a positive integer. Hence the rest of the "proof" is invalidated.

4. The proof picks the wrong cube root of 1. Choosing either non-real root gives us $0=0$. (Note: multiplying by x might be a tempting answer, but that in itself is perfectly valid. It's picking the wrong cube root that comes as a result of this which is wrong.)

5. Allowing for a leading digit to be zero, K is a four-digit number. Why? When 14 is divided by 13, we get a remainder of 1, so we need to add a number to 14 which, when divided by 13, gives remainder $13-1=12$. This number obviously ends in 00, so that when added to 14 its last two digits are 14. We thus need to consider 100, 200, 300, etc. Here's where you stop and think for a second: the remainders of ${}^{100}\!/_{13}$, ${}^{200}\!/_{13}$, ... are all different numbers, so the number we need to add to 14 will be less than 1300. Use the hint to "start at the top": check 1200, 1100, 1000, etc. We see that ${}^{1000}\!/_{13}$ gives remainder 12, so $K=1000+14=1014$. $1+0+1+4=6$.

6. One child is a boy born on a Tuesday. Suppose the first child is the boy born on Tuesday. Then the second child can either be a boy or a girl, born on any day of the week. There are seven possible days for the boy to be born and seven possible days for the girl to be born, so we have 14 total possibilities if the first child is a boy born on a Tuesday. The analysis is similar if we suppose the second child is the boy born on Tuesday, so there are 14 more possibilities. However, in counting this way, we're double counting the case where both were boys born on Tuesdays. So we have

$14-1=13$ ways for both children to be boys out of $14+14-1=27$ total possibilities, giving us a probability of $13/27$.

7. The person could be a local or a tourist, so the probability is $2/3 \cdot 3/4 + 1/3 \cdot 0 = 1/2$.

8. Since the answer changes, we know this person couldn't be a tourist and must therefore be a local. So the probability the airport is east is just the probability the local tells the truth the first three times and lies the last time, conditional on him giving one response three times in a row and a differing response the last time. The probability the airport is east is $\dfrac{\left(3/4\right)^3\left(1/4\right)}{\left(3/4\right)^3\left(1/4\right)+\left(1/4\right)^3\left(3/4\right)} = \dfrac{27/256}{30/256} = \dfrac{9}{10}$.

9. Only bulldogs have learned to follow the "sit" command. Since at least two dogs have learned to follow "sit" but not "roll over" and at least one has learned to follow both commands, we have at least three bulldogs. Since we have at least one of each dog, we have at least one golden retriever as well. So we have at most four poodles; i.e., we have at most four female dogs (since all the female dogs are poodles). Hence, we have at least four male dogs, and so we cannot have more females than males.

10. Since only bulldogs have learned "sit", poodles and retrievers must have learned "roll over". However, it doesn't say that *all* bulldogs have learned "sit", just that the only dogs who have learned "sit" are bulldogs. We could have a bulldog who learned "roll over" but not "sit".

11. I is true—simply write $\vec{x} = \left(\vec{x}-\vec{y}\right)+\vec{y}$. Then, by Triangle Inequality, $\left\|\vec{x}\right\| \le \left\|\vec{x}-\vec{y}\right\| + \left\|\vec{y}\right\|$, then subtract $\left\|\vec{y}\right\|$ from both sides to get the desired result. While II is a true statement (the famous Cauchy-Schwarz Inequality), it cannot be derived from the Triangle Inequality; in fact, it's actually used to prove the Triangle Inequality. III is also a true statement, but again it cannot be proved using the Triangle Inequality along; without any other conditions, it is impossible to establish equality. We can only show that $a+b \ge c$, $a+c \ge b$, and $b+c \ge a$. IV is true and follows through proof by induction. Rough sketch: apply the Triangle Inequality repeatedly to show that the straight line is shorter than any polygonal segment, then induct on the number of sides (adding one more side each time enables use of the inequality); as this limit goes to infinity, it is still inside an arc connecting the two points and so there is no shorter curved segment. Therefore, I and IV are the only true statements to follow directly from the Triangle Inequality.

12. The basis case is true but the inductive step cannot be validly used in the basis case. Why? Because if we split a set of one horse into two disjoint sets, one set is the empty set. There's no overlap between the empty set and the set containing the single horse, so the statement about an overlap and everything that follows from it is not supported by the basis case.

13. Without the assumption that $1<\sqrt{2}$, we can't assume that $n<m$. Note that the other facts are all true, but are implied by algebraic manipulation of $n<m$.

14. The time taken will be minimized if we can send the two slowpokes across together and not need either of them to cross again. So consider the following: A and B cross (3 minutes); A crosses back (2 minutes); C and D cross (8 minutes); B crosses back (3 minutes); A and B cross (3 minutes). The total amount of time is $3+2+8+3+3$ $=19$

15. First, consider the statements C makes to A. C says to A that $c=a+10$. If C were younger than A, that would be a lie, but C must tell the truth. Therefore, C is older than A (just not ten years older); therefore, we know $a<c$. C also says to A that $b<d$, but we just established that $c>a$, so this is a lie; therefore, $b>d$. This implies that D is telling the truth to B, that $d=e+9$. Since E is younger than D, E is also younger than B, so E's statement to B, that $e=a+7$, is true.

Now, B says to C that $c>d$ (and hence $c>e$) since C and E cannot be the same age. If $b>c$, this would be a lie, implying $e>c$, implying $a<c<e$. Then $c<d$ also, implying C's statement to D, that $|c-d|=6$, would be true; since $c<d$, we would have $c=d-6$. We also know that $e=d-9$, so it must have been the case that $e<c$, a contradiction. Therefore, $b<c$.

Thus, we know that $a<e<d<b<c$. Further, $e=a+7$, which implies $d=a+16$, and we know from A's statement to B that $b=\frac{17}{10}a$. From B's statement to C, $c-d=d-e$, which implies $c=2d-e=a+25$. Now since all ages are integers and $b=\frac{17}{10}a$, a must be a multiple of 10. Since $d<b<c$, $a+16<\frac{17}{10}a<a+25$, which implies $16<\frac{7}{10}a<25$, and the only multiple of 10 that a could be and fit this inequality is $a=30$. Using this, $b=51$, $c=55$, $d=46$, and $e=37$. Therefore, $a(c-b)+d+e=30(55-51)+46+37=203$.