For all questions, answer choice "E) NOTA" means none of the above answers is correct.

1. Suppose you have n statements A_1, A_2, ..., A_n. Define a "simple proof" to be a proof which, assuming exactly one of these statements to be necessarily true, proves that exactly one other statement is true (i.e., "If A_2, then A_5." would be a simple proof, but "If A_2, then A_4, A_5, and A_6" and "If A_2 and A_3, then A_5." would not be simple proofs). To prove the equivalence of all n statements using only simple proofs, what is the minimum number of simple proofs you would need to write?

A) n B) $2n$ C) $\frac{n(n-1)}{2}$ D) $(n-1)^2$ E) NOTA

For questions 2 and 3, consider the following "proof" that all positive integers are equal:

- It suffices to show that for any two positive integers x and y, $x=y$. Further, we may show this by showing that for all integers $N>0$, if the positive intgers x and y satisfy $\max(x,y)=N$, then $x=y$.
- Basis case: $N=1$. Here, if $\max(x,y)=1$ and x and y are positive integers, then the only possible case for both x and y is 1; so our basis case holds.
- Next, assume that the theorem is true for some positive integer k. Take x and y such that $\max(x,y)=k+1$. Then $\max(x-1,y-1)=k$; but by our hypothesis, this means that $x-1=y-1$. Adding 1 to both sides gives us $x=y$, and the theorem holds!

2. What type of proof is this?

A) direct B) indirect C) inductive D) deductive E) NOTA

3. Obviously, there must be a flaw in this proof. What is the *first* flaw?

A) Reducing the problem to showing that for all integers $N>0$, if x and y satisfy $\max(x,y)=N$, then $x=y$
B) Just showing that two arbitrary numbers x and y are equal
C) $\max(x-1,y-1)=k$ implies $x-1=y-1$
D) $\{\max(x-1,y-1)=k \text{ implies } x-1=y-1\}$ implies $x=y$
E) NOTA

4. Here's a proof that $3=0$: consider the equation $x^2+x+1=0$. Clearly, $x=0$ is not a solution; so, dividing through by x, this equation also satisfies $x+1+\frac{1}{x}=0$. By the original equation, $x+1=-x^2$, so we can write $-x^2+\frac{1}{x}=0$. We can then rewrite this as $x^2=\frac{1}{x}$. Again, since $x=0$ is not a solution, we can multiply through by x to give $x^3=1$. Therefore, $x=1$. Plugging $x=1$ into the original equation gives $1^2+1+1=0$. Hence, $3=0$.

What's wrong with this proof?

A) dividing equation by x
B) illegal substitution of $-x^2$ for $x+1$
C) multiplying equation by x
D) choosing the wrong value of x in the equation
E) NOTA

5. This question is inspired by pigeons: let K be the smallest positive integer divisible by 13 such that its last two digits are 14. Find the sum of the digits of K. (Hint: to minimize guesswork, "start at the top".)

A) 6 B) 8 C) 9 D) 12 E) NOTA

6. I have two children. One is a boy born on a Tuesday. What is the probability I have two boys?

A) $13/27$ B) $1/2$ C) $14/27$ D) $2/3$ E) NOTA

Use the following scenario for questions 7 and 8: you're lost in the faraway land of Houston. There are two types of people in Houston: tourists and locals. If you ask a local whether the airport is east or west of your current location, the probability you will receive a correct response is $3/4$. Tourists, however, are hopelessly lost and will always give you the wrong answer. Two-thirds of the population are locals. You have no way to tell, however, whether a person is a local or a tourist. Finally, you can ask the same person for directions multiple times and their responses will be independent of past responses (i.e., the probability a local tells you "east" the second time you ask for directions to the airport is independent of what he or she said the first time you asked; counterintuitive though it may seem, they are answering at random.

Armed with this knowledge:

7. You ask the first person you see whether the airport is east or west. He says "east". What is the probability the airport is indeed east?

A) $\frac{1}{4}$ B) $\frac{1}{2}$ C) $\frac{2}{3}$ D) $\frac{3}{4}$ E) NOTA

8. You ask this person three more times (i.e., a total of four times), and he says "east" the first three times and "west" the last time. What is the probability the airport is actually east?

A) $\frac{9}{128}$ B) $\frac{27}{70}$ C) $\frac{27}{41}$ D) $\frac{9}{10}$ E) NOTA

Use the following information for questions 9 and 10: eight dogs are enrolled in a class learning to follow two commands: "sit" and "roll over". Each dog is a bulldog, a golden retriever, or a poodle, and there is at least one of each breed present. All dogs are either male or female. All female dogs in the group are poodles. Currently:

- At least two of the dogs have learned to "sit" but not to "roll over".
- At least two of the dogs have learned to "roll over" but not to "sit".
- At least one dog has learned to follow both commands.
- Only bulldogs have learned to follow the "sit" command.

9. Which of the following statements cannot be true?

A) The class has more poodles than bulldogs
B) More dogs have learned to "sit" than to "roll over".
C) More dogs have learned to "roll over" than to "sit".
D) The class includes more females than males.
E) NOTA

10. If each dog has learned to follow at least one command, which of the following statements, if any, is not necessarily true?

A) All retrievers have learned to "roll over".
B) All poodles have learned to "roll over".
C) All bulldogs have learned to "sit".
D) No poodle has learned to "sit".
E) NOTA

11. The Triangle Inequality states that for any two vectors $\vec{x}$ and $\vec{y}$, $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$ (where $\|\bullet\|$ denotes the magnitude of a vector). Which of the following statements can you prove using only the Triangle Inequality? (In all places, where applicable, $\vec{x}$ and $\vec{y}$ are arbitrary vectors of equal dimension.)

I) $\|\vec{x}\| - \|\vec{y}\| \le \|\vec{x}-\vec{y}\|$ II) $|\vec{x}\cdot\vec{y}| \le \|\vec{x}\|\|\vec{y}\|$

III) In a triangle with side lengths a, b, and c, $c^2 = a^2 + b^2 - 2ab\cos C$, where C is the angle opposite side c

IV) The shortest path between two points is a straight line

A) I & II only B) II & III only C) I, III, & IV only D) I, II, III, & IV E) NOTA

12. Here's an inductive proof that all horses are the same color:

- Basis: If there is only one horse, there is only one color.
- Inductive: Now suppose the statement holds true for n horses, and consider a set of $n+1$ horses. Number the horses $1,\ 2,\ \ldots,\ n+1$. Now consider the sets $\{1,2,\ldots,n\}$ and $\{2,3,\ldots,n+1\}$. Each is a set of n horses, and so within each there is only one color. But since there is overlap in the sets and each horse is obviously the same color as itself, there must be only one color within the $n+1$ horses; i.e., all $n+1$ horses are the same color, and the theorem is proved.

Clearly there is a flaw here. What is it?

A) Basis case is false B) The inductive step cannot be validly used in the basis case

C) Large set cannot be broken into smaller sets in the inductive step

D) Cannot suppose that the statement holds true for n horses

E) NOTA

13. Several proofs exist which show that $\sqrt{2}$ is irrational. Here is one that proceeds by contradiction:

- Assume that $\sqrt{2}$ is a rational number. Then there exist positive integers m and n with $n \neq 0$ such that $\frac{m}{n} = \sqrt{2}$.
- Assume that $\frac{m}{n}$ is in lowest terms.
- Then $\sqrt{2} = \frac{m}{n} = \frac{m\left(\sqrt{2}-1\right)}{n\left(\sqrt{2}-1\right)} = \frac{2n-m}{m-n}$.
- Next, observe that $n < m$ implies $2n - m < m$.

- $\frac{2n-m}{m-n}=\frac{m}{n}$. Since $2n-m<m$, we also have that $m-n<n$, and so the fraction $\frac{m}{n}$ is not actually in simplest terms. Therefore, we have a contradiction, and so the assumption that $\sqrt{2}$ is rational must be false.

This proof is valid, but what key unstated (and unimplied) fact does the fourth bullet point utilize?

A) $1<\sqrt{2}$ B) $n<m$ C) $2n-m>m-n$ D) $m<2n$ E) NOTA

14. Four people—A, B, C, and D—need to cross a bridge late at night. The bridge is unstable, and so only up to two people can cross at one time; also, a torch must be carried by at least one of those crossing the bridge each time. Unfortunately, they only have one torch between them (so at least one person has to go back each time two people cross to the other side). A takes 2 minutes to cross the bridge, B takes three minutes, C takes 7 minutes, and D takes 8 minutes. When two people cross, they cross at the rate of the slower person. What is the least amount of time, in minutes, it will take for everyone to get across?

A) 17 B) 19 C) 20 D) 22 E) NOTA

15. Five people—A, B, C, D, and E—who share the same birthday start discussing their ages. Whenever a person is speaking to someone who is younger, he or she tells a lie; whenever a person is speaking to someone who is older, he or she tells the truth. No two people are the same age. The following statements are made (all ages are integers):

- D says to B, "I'm nine years older than E."
- E says to B, "I'm seven years older than A."
- A says to B, "Your age is exactly 70% greater than mine."
- B says to C, "E is younger than you."
- C says to D, "The difference between our ages is six years."
- C says to A, "I'm ten years older than you."
- C says to A, "B is younger than D."
- B says to C, "The difference between your age and D's age is the same as the difference between D's age and E's age."

Given these statements, let person A have age a, person B have age b, etc. Determine the value of $a(c-b)+d+e$.

A) -37 B) 152 C) 203 D) 341 E) NOTA