Solutions

1 **Answer: (C)** There are 1006 diameters of this circle. Once we choose a diameter, we can choose any of the remaining points and we will have a right triangle. Hence, the answer is:

$$\frac{1006 \cdot 2010}{\binom{2012}{3}} = \frac{3}{2011}.$$

Our desired sum is thus $3 + 2011 = 2014$, and we are done.

2 **Answer: (B)** The minimum value of u is clearly 0. The equation shown is a circle which lies entirely in the third quadrant (except at the points where it is tangent to the x and y axes). Because x and y are always negative, the product of the two is always positive except for the points at which either x or y is zero. On the other hand, if we want to maximize xy, we choose $x = y$. Hence, we have $x^2+2x+1+x^2+2x = 0$, which implies that $x = \frac{-4\pm2\sqrt{2}}{2}$. To maximize xy, we choose the negative solution. Thus, $xy = \left(\frac{-2-\sqrt{2}}{2}\right)^2 = \frac{3}{2}+\sqrt{2}$. The sum we require is

$$\frac{3}{2} + \sqrt{2} + 0 = \frac{3}{2} + \sqrt{2}.$$

3 **Answer: (B)** Let the slope of one of the lines be a. We know the tangent line can be expressed in the form $y - 3 = a(x - 5)$, which implies that $ax - y = 5a - 3$. The distance between this line and the center of the circle should equal the radius of the circle, 1. Mathematically,

$$\frac{|-6a+4|}{\sqrt{a^2+1}} = 1 \implies a = \frac{24 \pm \sqrt{51}}{35}$$

The two solutions, are, of course, the two slopes mentioned in the question. We want $a = \frac{24+\sqrt{51}}{35}$ which makes our sum $24 + 51 + 35 = 110$.

4 **Answer: (D)** The area of the triangle is half the magnitude of

$$\Big(1-0, 1-0, 2-0\Big) \times \Big(3-1, 4-1, 2-2\Big)$$

where the above two vectors are from the origin to the other two points. This cross product is: $< -6, 4, 1 >$ which has magnitude $\sqrt{53}$ giving area of $\sqrt{53}/2$.

5 **Answer: (B)** $1 \cdot 7 + 1 \cdot k = 0 \implies k = -7$

6 **Answer (D):** This is simply one more than the nth triangular number, or $\dfrac{n^2+n+2}{2}$. Plugging in 2011 gives the desired answer choice.

7 **Answer (E):** The sum of the slopes of a hyperbola is always 0, regardless of its equation. The probability is thus 1.

8 **Answer: (B)** We want k such that

$$\frac{3-1}{k-1} = \frac{k-2}{2-k}.$$

Solving this for k, we find $k = 1 \pm \sqrt{2}$. There are two such values of k, and hence our desired sum is

$$1 + \sqrt{2} + 1 - \sqrt{2} + 2 = 4,$$

as desired.

9 **Answer: (A)** The cross product finds a vector which is perpendicular to the other two. Because $\vec{i}$ and $\vec{k}$ essentially represent the y and z axes, $\vec{j} \times \vec{k} = \vec{i}$. Anything crossed with itself is $\vec{0}$ (which is different than zero!). The magnitude of the zero vector is 0.

10 **Answer: (A)** As before, $j \times k = i$, and $j \times i = k$, and $k \times k = \vec{0}$. We can stop here because anything crossed with the zero vector is the zero vector. Thus, $\vec{u}$ is the zero vector as well. Its magnitude is clearly zero.

11 **Answer (D):** Simply use the distance formula between the centers and set this equal to the sum of the radi. Two values are possible, 3 and 7. Their sum is 10.

12 **Answer: (D)** This property can be realized by letting two vectors have coordinates $\langle a_1, a_2, a_3, \rangle$, and $\langle b_1, b_2, b_3\rangle$. Then:

$$\begin{aligned}
|a \times b|^2 &= (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 \\
&\quad + (a_1b_2 - a_2b_1)^2 \\
&= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) \\
&\quad - (a_1b_1 + a_2b_2 + a_3b_3)^2 \\
&= |a|^2|b|^2 - (a \cdot b)^2 \\
&= |a|^2|b|^2 - |a|^2|b|^2 \cos^2\theta \\
&= |a|^2|b|^2(1 - \cos^2\theta)
\end{aligned}$$

Taking the square root then gives the answer. Note that we can take the square root because $0 \le \theta \le \pi$, and $\sin\theta$ is always positive on this interval.

13 **Answer: (A)** Clearly,

$$\cos\theta = \frac{9}{\sqrt{3} \cdot \sqrt{29}} = \frac{3\sqrt{87}}{29} \approx \frac{9}{10}$$

Because $\cos\alpha = 1$ when $\alpha = 0$, θ must be relatively close to 0. Clearly, it is in the interval $(0, 20)$.

14 **Answer: (D)** Consider the vectors $\overline{AB} = \langle -1, 1, 0\rangle$ and $\overline{AC} = \langle 1, 3, 6\rangle$. The plane which contains these points is perpendicular to the vector $\overline{AB} \times \overline{AC} = \langle 6, 6, -4\rangle$. The equation of the plane is thus: $6x + 6y - 4z = 14$. The distance between the given point and the plane is:

$$\frac{|-1\cdot 6 - 2\cdot 6 + 2\cdot -4 - 14|}{\sqrt{36+36+16}} = \frac{10\sqrt{22}}{11},$$

as desired.

15 **Answer: (B)** By definition, $\cos\theta = 0$ means the vectors are orthogonal.

16 **Answer: (A)** We have $3r + 2x = 5$, which implies that $5x^2 + 20x + 9y^2 - 25 = 0$, or, in standard form:

$$\frac{(x+2)^2}{9} + y^2/5 = 1.$$

The area desired is: $\pi \cdot 3 \cdot \sqrt{5} = 3\pi\sqrt{5}$.

17 **Answer: (B)** This is a parabola, and parabolas always have eccentricity of 1.

18 **Answer (D):** The sum of the distances from any point on an ellipse to the two foci of the ellipse is always a constant, which is equal in magnitude to the length of the major axis. Thus, if the point $P = (0, 0)$ lies on the ellipse, and $F = (2012, 2012)$ is one of the foci, then we have that $PF = 2012\sqrt{2}$. We then seek another point such, Q, such that $PQ = 2012^2\sqrt{2} - 2012\sqrt{2} = 2012\sqrt{2}\cdot(2011)$. The only point listed which satisfies is answer choice (D).

19 **Answer: (C)** When $x = 2$, $y = 3 \pm \sqrt{2}$. Our desired point(s) are $(2, 3\pm\sqrt{2})$. The slope between the points $(2, 3+\sqrt{2})$ and $(2, 3)$ is perpendicular to the slope of the tangent. Thus, we want:

$$-\left(\frac{2-2}{3\pm\sqrt{2}-3}\right) = 0$$

Note that this value is independent of which point we choose.

20 **Answer: (A)** Our conversion is accomplished by:

$$\left(2\cdot\cos\frac{\pi}{3}, 2\cdot\sin\frac{\pi}{3}\right) = (1, \sqrt{3})$$

21 **Answer: (B)** Our answer is given by: $\dfrac{8!}{2!} = 20160$.

22 **Answer: (B)** If the three points given are to form an equilateral triangle, then the point $(5, 3)$, when rotated $60°$, must equal (a, b). Note that a counterclockwise rotation will give a point in the first quadrant, where both the x and y coordinates are positive. If we rotate clockwise, we get a point in the 4th quadrant, and differing signed among a and b. This case, then, will produce the smallest product. Hence, we have:

$$\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2+\sqrt{3} \\ 1-2\sqrt{3} \end{pmatrix}$$

Because we set $(1, 1)$ to be our "origin" we must add 1 to each of the x and y coordinates. Thus,

$$(a, b) = (3+\sqrt{3}, 2-2\sqrt{3}),$$

and the product is:

$$\begin{aligned}(3+\sqrt{3})(-2\sqrt{3}+2) &= -6\sqrt{3}+6-6+2\sqrt{3} \\ &= -4\sqrt{3},\end{aligned}$$

as desired. **Note:** we rotated the point $(4, 2)$ instead of $(5, 3)$ because the rotational matrix rotates ABOUT the origin, and we needed essentially treated $(1, 1)$ as the origin ($(4, 2)$ is the relative position of $(5, 3)$ if $(1, 1)$ is the origin).

23 **Answer: (E)** If the side of the cube has length a, we can say one of the face diagonals is the vector $\langle a, a, 0\rangle = \vec{S}$ and the space diagonal is simply $\langle a, a, a\rangle = \vec{F}$. Then,

$$\frac{\vec{F}\cdot\vec{S}}{||\vec{F}|\cdot|||\vec{S}||} = \frac{a^2+a^2}{\sqrt{2a^2}\cdot\sqrt{3a^2}} = \frac{2}{\sqrt{6}} = \cos\theta.$$

Whence, $\theta = \arccos\left(\frac{\sqrt{6}}{3}\right)$.

24 **Answer: (B)**

25 **Answer: (E)** We have: $2\sin t\cos t = x$ which implies that $2y\sqrt{1-y^2} = x$ or $4y^4 - 4y^2 - x^2 = 0$. This represents none of the conic sections listed among the answer choices.

26 **Answer: (D)** There are 12 petals on the graph, each with area of 6π, and hence the answer is 72π.

27 **Answer: (D)** We wish to solve the system of equations:

$$\begin{aligned}1+1+A+B+C &= 0 \\ 0+4+2B+C &= 0 \\ 16+0+4A+C &= 0\end{aligned}$$

Doing so gives: $A = -8$, $B = -10$, and $C = 16$, the product of which is 1280.

28 **Answer: (A)** The cicum-radius of this triangle is 1 (the magnitude of each of the roots).This means each side is $\sqrt{3}$, and our area is:

$$\frac{\left(\sqrt{3}\right)^2\sqrt{3}}{4} = \frac{3\sqrt{3}}{4},$$

as desired.

29 **Answer: (A)** Both the answer choices and the expression remind the reader of polar graphs. Indeed, we have:

$$r^3 = 2r^2 \cos\theta \sin\theta,$$

or, more simply, $r = 2\cos\theta\sin\theta$, which is the 4 petaled rose $r = \sin(2\theta)$.

30 **Answer: (D)** Trivially,

$$C = \left(\frac{1+2+4}{3}, \frac{1+3+7}{3}\right) = \left(\frac{7}{3}, \frac{11}{3}\right)$$

The desired sum is thus $\frac{7+11}{3} = 6$.