2012 Nationals Integration Solutions

1. E The correct answer is $x^3 + C$
2. A Evaluate $e^x|_{-10}^{10}$ gives the correct area.
3. A Use the shell method:

$$2\pi \int_0^3 x(54 - 2x^3)dx = 2\pi * \left(27x^2 - \frac{2x^5}{5}\right)\Big|_0^3 = \frac{1458\pi}{5}$$

4. D It is an odd function, which is symmetric across the origin. The area from -3 to 3 is 0, and therefore, the average value is 0.
5. D Substitute $1/x$ for t to get

$$\frac{\left(\frac{1}{x}\right)^2 + \sin\frac{\pi}{x}}{1 - \frac{1}{x}} d\left(\frac{1}{x}\right) = \frac{\left(\frac{1}{x^2} + \sin\frac{\pi}{x}\right)}{\frac{x-1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{\frac{1}{x^2} + \sin\frac{\pi}{x}}{x^2 - x}$$

6. A The particle changes direction at $t = 2$ so the distance must be evaluated as two different integrals.

$$\left|\int_1^2 \left(\frac{2}{t} - 1\right) dt\right| + \left|\int_2^4 \left(\frac{2}{t} - 1\right) dt\right| = |2\ln 2 - 2| + |2\ln 2 - 1|$$
$$= 2 - 2\ln 2 + 2\ln 2 - 1 = 1$$

7. A Factor a negative and it is the derivative of $\csc x$.

$$-\csc x\,|_{\pi/4}^{\pi/2} = -\left(1 - \sqrt{2}\right) = \sqrt{2} - 1$$

8. B The given equation is a circle with a radius of 1. Thus, the area is π. Alternatively,

$$\frac{1}{2}\int_0^\pi r^2 d\theta = \frac{1}{2}\int_0^\pi 4\sin^2\theta\, d\theta = \theta - \frac{\sin 2\theta}{2}\Big|_0^\pi = \pi$$

9. B Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}dx, dx = 2\sqrt{x}du = 2udu$. Then you can integrate by parts.

$$2\int_1^3 ue^u du = 2(ue^u - e^u|_1^3) = 2 * 2e^3 = 4e^3$$

10. E Simpson's Rule cannot be used with an odd number of subintervals.
11. E The upper bound oscillates; therefore, no actual limit exists.
12. C Using the shell method

$$2\pi \int_0^1 x * x^{2011} dx = 2\pi \frac{x^{2013}}{2013}\Big|_0^1 = \frac{2\pi}{2013}$$

13. E The integral diverges at $x = 1$, and cannot be evaluated.

14. B You can rearrange the equation to be $\ln x - 2x + 1 = y'$. Integrating both sides result in $x \ln x - x + x - x^2 = y$, $y = x \ln x - x^2$.

15. A Using trig identities, we can convert it to

$$\int_{\frac{-\pi}{6}}^{\frac{\pi}{12}} \tan 2x \, dx = \int_{\frac{-\pi}{6}}^{\frac{\pi}{12}} \frac{\sin 2x}{\cos 2x} dx = \int_{\frac{-\pi}{6}}^{\frac{\pi}{12}} \frac{2 \sin x \cos x}{2\cos^2 x - 1} dx = -\frac{1}{2} \ln|2\cos^2 x - 1| \Big|_{-\pi/6}^{\pi/12} = -\frac{\ln 3}{4}$$

16. A Graphing the equation will make it clear that the region is horizontally symmetrical, and the horizontal centroid will thus be in the middle of region. The two x intercepts are 1 and 2, so the center is at 3/2.

17. B Multiply the numerator and denominator by $1/n^2$ to get

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{\frac{2}{n}}{1 + \frac{i^2}{n^2}} = \int_0^1 \frac{2}{1 + x^2} dx = 2\arctan x|_0^1 = \frac{\pi}{2}$$

18. C The area of the region above the curve $f(x) = x^3 - 9x$ is found by

$$\left| \int_0^3 (9x - x^3) dx \right| = \frac{81}{4}$$

The area that is not also above the line $f(x) = -8x$ is found by

$$\left| \int_0^1 (x^3 - 9x + 8x) dx \right| = \frac{1}{4}$$

So the final probability is

$$\frac{81 - 1}{81} = \frac{80}{81}$$

19. C Using integration by parts

$$\int e^x \cos x \, dx = \cos x \, e^x + \int \sin x \, e^x dx$$

$$\int \sin x \, e^x dx = \sin x \, e^x - \int \cos x \, e^x dx$$

$$\int e^x \cos x \, dx = \cos x \, e^x + \sin x \, e^x - \int \cos x \, e^x dx$$

$$2\int e^x \cos x \ dx = \cos x \, e^x + \sin x \, e^x$$

$$\int e^x \cos x \ dx = \frac{1}{2} e^x(\cos x + \sin x) + C$$

20. B Integrating through each cross section:

$$\frac{\sqrt{3}}{4}\int_0^4 \left(\frac{1}{2}x^2\right)^2 dx \ = \frac{\sqrt{3}}{16} * \frac{x^5}{5}\Big|_0^4 = \frac{64\sqrt{3}}{5}$$

21. B Solve $y = \frac{1}{2}x^2$ for $x = \sqrt{2y}$. Twice this length makes up each side of the triangles

$$\frac{\sqrt{3}}{4}\int_0^8 \left(2\sqrt{2y}\right)^2 dy = \sqrt{3}\int_0^8 2y dy = 64\sqrt{3}$$

22. A Complete the square and substitute $u = x - 5$:

$$\int_{11}^{4\sqrt{3}+5} \frac{dx}{(x-5)\sqrt{x^2-10x-11}} = \int_{11}^{4\sqrt{3}+5} \frac{dx}{(x-5)\sqrt{(x-5)^2-36}} = \int_6^{4\sqrt{3}} \frac{du}{u\sqrt{u^2-36}}$$

$$= \frac{1}{6}\operatorname{arcsec}\frac{u}{6}\Big|_6^{4\sqrt{3}} = \frac{\pi}{36}$$

23. B For a finite n, the sequence is bounded below by

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \int_n^{2n} \frac{dx}{x} = \ln 2$$

By moving the first term to the other side, the sequence is now bounded above by

$$\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \int_n^{2n} \frac{dx}{x} + \frac{1}{n} = \ln 2 + \frac{1}{n}$$

As $n \to \infty, 1/n,$ the size of the interval that contains the series decreases to 0. Thus, the sequence must approach $\ln 2$.

24. D Wallis's Formulas (derived from typical techniques for trig integrals) state that if n is odd, then

$$\int_0^{\pi/2} \cos^n x\, dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right) \dots \left(\frac{n-1}{n}\right)$$

For $n = 7, \frac{2}{3} * \frac{4}{5} * \frac{6}{7} = \frac{16}{35}$

25. E Since there are three petals, our bounds are $-\frac{\pi}{6}$ and $\frac{\pi}{6}$.

$$A = \frac{1}{2}\int_{-\pi/6}^{\pi/6} (\pi \cos 3\theta)^2 d\theta = \frac{\pi^2}{2}\int_{-\pi/6}^{\pi/6} \left(\frac{1}{2} + \frac{\cos 6\theta}{2}\right) d\theta = \frac{\pi^2}{2}\left(\frac{x}{2} + \frac{\sin 6\theta}{12}\right)\Big|_{-\pi/6}^{\frac{\pi}{6}} = \frac{\pi^3}{12}$$

26. D Substitute $u = x^2, du = 2xdx$

$$\int_0^5 x^3 e^{x^2} dx = \int_0^{25} \frac{1}{2} u e^u du = 12e^{25} + \frac{1}{2}$$

27. D With a little simplification, the summation becomes:

$$\lim_{n\to\infty} \sum_{i=0}^{n} \frac{\frac{5}{n}\left(3\left(\frac{5i}{n} + 2\right)^2 + \frac{5i}{n} + 2\right)^2}{\frac{5i}{n} + 2 + 1}$$

The width of the integral is 5, as seen by $\frac{5}{n}$ and $\frac{5i}{n}$. The integral begins at $x = 2$ and ends 5 units later, at $x = 7$. Treating $\frac{5i}{n} + 2$ as the increment and $\frac{5}{n}$ as dx, we get the integral

$$\int_2^7 \frac{(3x^2 + x)^2}{x + 1} dx$$

28. B Leaving a $\sec^2 x$ to be the dx for the tangents, change the remaining $\sec x$ into $\tan x$

$$\int_0^{\pi/4} \sec^2 x * \tan^4 x * \sec^4 x\, dx = \int_0^{\pi/4} \sec^2 x * \tan^4 x * (\tan^4 x + 2\tan^2 x + 1)$$

$$= \int_0^{\pi/4} \sec^2 x\, (\tan^8 x + 2\tan^6 x + \tan^4 x) = \frac{188}{315}$$

29. C To find the inverse of $y = x^2 - 4x + 3$, we can use the quadratic formula, to solve for x when the equation is equal to 0. $x^2 - 4x + 3 - y = 0$. In this case, y is a constant.

$$x = \frac{4 \pm \sqrt{16 - 4(3 - y)}}{2} = 2 \pm \sqrt{1 + y}$$

$$g(x) = 2 \pm \sqrt{1+x}$$

Only the positive case should be considered because $x > 2$. Then integration gives

$$\int_0^4 \left(2+\sqrt{1+x}\right)dx = 2x + \frac{2}{3}(1+x)^{\frac{3}{2}}\Big|_0^4 = \frac{22}{3} + \frac{10\sqrt{5}}{3}$$

30. C Complete the square in the denominator and let $u = x + \frac{1}{2}$

$$\int_{-1/2}^{1} \frac{x}{4x^2+4x+10}dx = \int_{-1/2}^{1} \frac{x}{4\left(x+\frac{1}{2}\right)^2+9}dx$$

$$= \int_0^{3/2} \frac{u-\frac{1}{2}}{4u^2+9}du = \int_0^{3/2} \frac{u}{4u^2+9}du - \frac{1}{2}\int_0^{3/2} \frac{du}{4u^2+9}$$

$$= \frac{1}{8}\ln|4u^2+9|\Big|_0^{3/2} - \frac{1}{12}\arctan\frac{2u}{3}\Big|_0^{3/2} = \frac{\ln 2}{8} - \frac{\pi}{48}$$