2012 Nationals Integration Solutions

1. E The correct answer is $x^3 + C$
2. A Evaluate $e^x\big|_{10}^0$ gives the correct area.
3. A Use the shell method:

$$2\pi \int_0^3 x(54 - 2x^3)\,dx = 2\pi \left(27x^2 - \frac{2x^5}{5}\right)\bigg|_0^3 = \frac{1458\pi}{5}$$

4. D It is an odd function, which is symmetric across the origin. The area from -3 to 3 is 0, and therefore, the average value is 0.
5. D Substitute $1/x$ for $t$ to get

$$\frac{\left(\frac{1}{x}\right)^2 + \sin \frac{\pi}{x} d\left(\frac{1}{x}\right)}{1 - \frac{1}{x}} = \left(\frac{1}{x^2} + \sin \frac{\pi}{x}\right)\left(-\frac{1}{x^2}\right) = \frac{1}{x^2} + \sin \frac{\pi}{x}$$

6. A The particle changes direction at $t = 2$ so the distance must be evaluated as two different integrals.

$$\left|\int_1^2 \left(\frac{2}{t} - 1\right)\,dt\right| + \left|\int_2^4 \left(\frac{2}{t} - 1\right)\,dt\right| = |2\ln 2 - 2| + |2\ln 2 - 1|$$

$$= 2 - 2\ln 2 + 2\ln 2 - 1 = 1$$

7. A Factor a negative and it is the derivative of $\csc x$.

$$-\csc x \big|_{\pi/4}^{\pi/2} = -(1 - \sqrt{2}) = \sqrt{2} - 1$$

8. B The given equation is a circle with a radius of 1. Thus, the area is $\pi$. Alternatively,

$$\frac{1}{2} \int_0^\pi r^2\,d\theta = \frac{1}{2} \int_0^\pi 4\sin^2 \theta\,d\theta = \theta - \frac{\sin 2\theta}{2}\bigg|_0^\pi = \pi$$

9. B Let $u = \sqrt{x}, du = \frac{1}{2\sqrt{x}}\,dx, dx = 2\sqrt{x}\,du = 2udu$. Then you can integrate by parts.

$$2 \int_1^3 ue^udu = 2(ue^u - e^u\big|_1^3) = 2 * 2e^3 = 4e^3$$

10. E Simpson’s Rule cannot be used with an odd number of subintervals.
11. E The upper bound oscillates; therefore, no actual limit exists.
12. C Using the shell method

$$2\pi \int_0^1 x * x^{2011}\,dx = 2\pi \left.\frac{x^{2013}}{2013}\right|_0^1 = \frac{2\pi}{2013}$$

13. E The integral diverges at $x = 1$, and cannot be evaluated.
14. B You can rearrange the equation to be \( \ln x - 2x + 1 = y' \). Integrating both sides result in \( x \ln x - x + x - x^2 = y \), \( y = x \ln x - x^2 \).

15. A Using trig identities, we can convert it to

\[
\int_{-\pi/6}^{\pi/12} \frac{\tan 2x}{\cos 2x} \, dx = \int_{-\pi/6}^{\pi/12} \sin 2x \, dx = \int_{-\pi/6}^{\pi/12} \frac{2 \sin x \cos x}{2 \cos^2 x - 1} \, dx = -\frac{1}{2} \ln \left| 2 \cos^2 x - 1 \right|_{-\pi/6}^{\pi/12} = -\frac{\ln 3}{4}
\]

16. A Graphing the equation will make it clear that the region is horizontally symmetrical, and the horizontal centroid will thus be in the middle of region. The two x intercepts are 1 and 2, so the center is at 3/2.

17. B Multiply the numerator and denominator by \( 1/n^2 \) to get

\[
\lim_{n \to \infty} \sum_{i=0}^{n} \frac{2}{1 + \frac{i^2}{n^2}} = \int_{0}^{1} \frac{2}{1 + x^2} \, dx = 2 \arctan x \bigg|_{0}^{1} = \frac{\pi}{2}
\]

18. C The area of the region above the curve \( f(x) = x^3 - 9x \) is found by

\[
\left| \int_{0}^{3} (9x - x^3) \, dx \right| = \frac{81}{4}
\]

The area that is not also above the line \( f(x) = -8x \) is found by

\[
\left| \int_{0}^{1} (x^3 - 9x + 8x) \, dx \right| = \frac{1}{4}
\]

So the final probability is

\[
\frac{81 - 1}{81} = \frac{80}{81}
\]

19. C Using integration by parts

\[
\int e^x \cos x \, dx = \cos x \cdot e^x + \int \sin x \cdot e^x \, dx
\]

\[
\int \sin x \cdot e^x \, dx = \sin x \cdot e^x - \int \cos x \cdot e^x \, dx
\]

\[
\int e^x \cos x \, dx = \cos x \cdot e^x + \sin x \cdot e^x - \int \cos x \cdot e^x \, dx
\]
20. B Integrating through each cross section:

\[ \frac{\sqrt{3}}{4} \int_0^4 \left( \frac{1}{2}x^2 \right)^2 dx = \frac{\sqrt{3}}{16} \left. \frac{x^5}{5} \right|_0^4 = \frac{64\sqrt{3}}{5} \]

21. B Solve \( y = \frac{1}{2}x^2 \) for \( x = \sqrt{2y} \). Twice this length makes up each side of the triangles

\[ \frac{\sqrt{3}}{4} \int_0^8 (2\sqrt{2y})^2 dy = \sqrt{3} \int_0^8 2y dy = 64\sqrt{3} \]

22. A Complete the square and substitute \( u = x - 5 \):

\[
\int_{11}^{4\sqrt{3}+5} \frac{dx}{(x-5)\sqrt{x^2-10x-11}} = \int_{11}^{4\sqrt{3}+5} \frac{dx}{(x-5)\sqrt{(x-5)^2-36}} = \int_6^{4\sqrt{3}} \frac{du}{u\sqrt{u^2-36}} = \frac{1}{6} \arccos \left. \frac{u}{6} \right|_6^{4\sqrt{3}} = \frac{\pi}{36}
\]

23. B For a finite \( n \), the sequence is bounded below by

\[
\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} > \int_n^{2n} \frac{dx}{x} = \ln 2
\]

By moving the first term to the other side, the sequence is now bounded above by

\[
\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n} < \int_n^{2n} \frac{dx}{x} + \frac{1}{n} = \ln 2 + \frac{1}{n}
\]

As \( n \to \infty \), the size of the interval that contains the series decreases to 0. Thus, the sequence must approach \( \ln 2 \).

24. D Wallis’s Formulas (derived from typical techniques for trig integrals) state that if \( n \) is odd, then
\[ \int_0^{\pi/2} \cos^n x \, dx = \left( \frac{2}{3} \right) \left( \frac{4}{5} \right) \cdots \left( \frac{n-1}{n} \right) \]

For \( n = 7 \), \( \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} = \frac{16}{35} \)

25. E Since there are three petals, our bounds are \(-\frac{\pi}{6}\) and \(\frac{\pi}{6}\).

\[
A = \frac{1}{2} \int_{-\pi/6}^{\pi/6} (\pi \cos 3\theta)^2 d\theta = \frac{\pi^2}{2} \int_{-\pi/6}^{\pi/6} \left( \frac{1}{2} + \frac{\cos 6\theta}{2} \right) d\theta = \frac{\pi^2}{2} \left( \frac{x}{2} + \frac{\sin 6\theta}{12} \right) \bigg|_{-\pi/6}^{\pi/6} = \frac{\pi^3}{12}
\]

26. D Substitute \( u = x^2 \), \( du = 2xdx \)

\[
\int_0^5 x^3 e^{x^2} \, dx = \int_0^{25} u \cdot e^u \, du = 12e^{25} + \frac{1}{2}
\]

27. D With a little simplification, the summation becomes:

\[
\lim_{n \to \infty} \sum_{i=0}^{n} \frac{5}{n} \left( \frac{3i}{n} + 2 \right)^2 \left( \frac{5i}{n} + 2 + 1 \right)
\]

The width of the integral is 5, as seen by \( \frac{5}{n} \) and \( \frac{5i}{n} \). The integral begins at \( x = 2 \) and ends 5 units later, at \( x = 7 \). Treating \( \frac{5i}{n} + 2 \) as the increment and \( \frac{5}{n} \) as \( dx \), we get the integral

\[
\int_2^7 \frac{(3x^2 + x)^2}{x + 1} \, dx
\]

28. B Leaving a \( \sec^2 x \) to be the \( dx \) for the tangents, change the remaining \( \sec x \) into \( \tan x \)

\[
\int_0^{\pi/4} \sec^2 x \cdot \tan^4 x \cdot \sec^4 x \, dx = \int_0^{\pi/4} \sec^2 x \cdot \tan^4 x \cdot (\tan^4 x + 2 \tan^2 x + 1)
= \int_0^{\pi/4} \sec^2 x (\tan^8 x + 2 \tan^6 x + \tan^4 x) = \frac{188}{315}
\]

29. C To find the inverse of \( y = x^2 - 4x + 3 \), we can use the quadratic formula, to solve for \( x \) when the equation is equal to 0. \( x^2 - 4x + 3 - y = 0 \). In this case, \( y \) is a constant.

\[
x = \frac{4 \pm \sqrt{16 - 4(3-y)}}{2} = 2 \pm \sqrt{1+y}
\]
\[ g(x) = 2 \pm \sqrt{1 + x} \]

Only the positive case should be considered because \( x > 2 \). Then integration gives

\[
\int_0^4 (2 + \sqrt{1 + x}) \, dx = 2x + \frac{2}{3} (1 + x) \frac{3}{2} \bigg|_0^4 = \frac{22}{3} + \frac{10\sqrt{5}}{3}
\]

30. C Complete the square in the denominator and let \( u = x + \frac{1}{2} \)

\[
\int_{-1/2}^1 \frac{x}{4x^2 + 4x + 10} \, dx = \int_{-1/2}^1 \frac{x}{4\left(x + \frac{1}{2}\right)^2 + 9} \, dx
\]

\[
= \int_0^{3/2} \frac{u - \frac{1}{2}}{4u^2 + 9} \, du = \int_0^{3/2} \frac{u}{4u^2 + 9} \, du - \frac{1}{2} \int_0^{3/2} \frac{du}{4u^2 + 9}
\]

\[
= \left. \frac{1}{8} \ln|4u^2 + 9| \right|_0^{3/2} - \frac{1}{12} \arctan \frac{2u^{3/2}}{3} \bigg|_0^{3/2} = \frac{\ln 2}{8} - \frac{\pi}{48}
\]