ANSWERS

(1) **DCDCB**	(6) **CDDAE**	(11) **BDABC**
(16) **DCBBA**	(21) **AADBD**	(26) **BCDCD**

SOLUTIONS

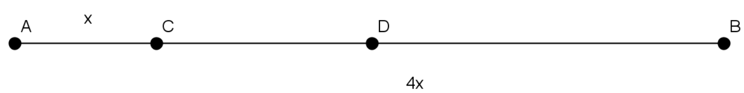
1. Noting that $x-2=(\sqrt{x}+\sqrt{2})(\sqrt{x}-\sqrt{2})$, we have $\dfrac{x-2}{\sqrt{x}-\sqrt{2}}=\sqrt{x}+\sqrt{2}$. Hence the area of the circle is $\pi(\sqrt{x}+\sqrt{2})^2=\pi(x+\sqrt{2x}+2)$, **D.**

2. The angles α_1 and α_7 are same-side exterior angles. Hence $\alpha_1+\alpha_7=180^\circ \Rightarrow \alpha_7=180^\circ-70^\circ=110^\circ$, **C.**

3. There are three possibilites we can have: $\alpha_1+\alpha_4$ or equivalent, $\alpha_1+\alpha_2$ or equivalent, and $\alpha_2+\alpha_3$ or equivalent. The highest of these is $\alpha_2+\alpha_3=110^\circ+110^\circ=220^\circ$, **D.**

4. $S>630^\circ$ only when α_k is acute. Exactly half of the α_i are acute; therefore our answer is $1/2$, **C.**

5. Consider the following diagram and let $AC=x$ and $BC=4x$.



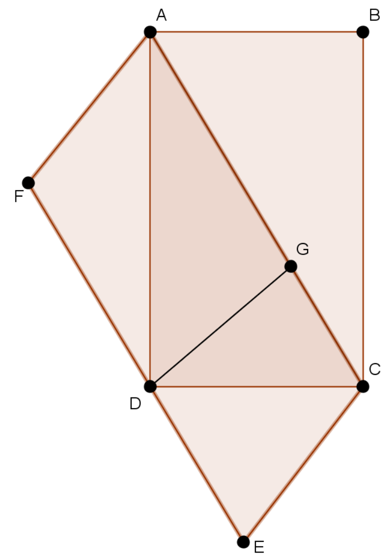
Then if D is chosen some distance a from C, we have $CD=a$ and $BD=4x-a$. If we can form a triangle with these three lengths, then by the Triangle Inequality we have two inequalities:

$$x+(4x-a)>a \Rightarrow a<\frac{5x}{2} \qquad (1)$$

$$x+a>4x-a \Rightarrow a>\frac{3x}{2} \qquad (2)$$

Since $\frac{3x}{2}<CD<\frac{5x}{2}$, and we can choose D from a total length of $5x-x=4x$, the probability we desire is $\dfrac{(\frac{5x}{2}-\frac{3x}{2})}{4x}=\dfrac{1}{4}$, **B.**

6. Choose point G on the hypotenuse of right triangle ADC such that $\overline{DG}$ is an altitude of the triangle.



Since $\overline{AC}||\overline{EF}$, $\overline{DG}$ is also perpendicular to $\overline{EF}$. Thus $\overline{DG}$ is the height of parallelogram $ACEF$, and we can write $[ACEF] = (DG)(AC)$. But since $[ABCD] = 2[ADC] = 2\left(\frac{(DG)(AC)}{2}\right) = (DG)(AC)$, we have $[ACEF] = [ABCD] = 96$, **C.**

7. A unit triangle is an equilateral triangle with an area of 1. Hence if its side length is s, we have $\frac{s^2\sqrt{3}}{4} = 1 \Rightarrow s^2 = \frac{4}{\sqrt{3}} \Rightarrow s = \frac{2}{\sqrt[4]{3}}$. Thus the perimeter is $p = 3s = \frac{6}{\sqrt[4]{3}}$, **D.**

8. As the number of sides of regular polygon increases, it approaches the shape of a circle. In the case of the unit polygons, the area of this circle is 1. Thus we want to find the circumference of a circle with area 1. We have $\pi r^2 = 1 \Rightarrow r = \frac{1}{\sqrt{\pi}}$ and the circumference is $2\pi r = 2\pi(\frac{1}{\sqrt{\pi}}) = 2\sqrt{\pi}$, **D.**

9. When $x \geq 0$, we have $f(x) = 2x$. However, when $x < 0$, we can let $x = -k$ for some $k > 0$. Then $f(x) = |x + |-k|| = |-k + k| = 0$. For $0 \leq x < 1$, $f(x)$ is a triangle with a height of $2(1) = 2$ and base of 1 and thus has an area of 1. For $-1 < x < 0$, $f(x)$ is a straight line along the x-axis and thus has no area. The total area is 1, **A.**

10. Note that a_n is the number of diagonals for a regular polygon when $n \geq 3$. Thus our answer is simply $a_z = 275$, **E.**

11. Cylic quadrilaterals have the property that opposite angles add up to $180°$. Hence since opposite angles in a rhombi are equal, in this case, they must both be $90°$. Furthermore since opposite sides in a rhombus are equal, this rhombus must be a square. Its area is simply $\frac{12^2}{2} = 72$, **B.**

12. The trisection points divide the triangle into a hexagon and three triangles. Each of these triangles are similar to T by a ratio of $\frac{1}{3}$. Thus the ratio of the area of each of these triangles to the area of T is $(\frac{1}{3})^2 = \frac{1}{9}$. Since there are three triangles, they account for $3(\frac{1}{9}) = \frac{1}{3}$ of the total area of T. Thus the ratio of the area of the hexagon to the area of the entire triangle is $1-3\left(\dfrac{1}{9}\right) = \dfrac{2}{3}$, **D.**

13. Let the center be (x, y). Thus the distance from the center to the three points is equal; using distance formula, we have

$$\begin{aligned}(x-2)^2+(y-5)^2 = x^2+(y+1)^2 \quad &\Rightarrow \quad x^2-4x+4+y^2-10y+25 = x^2+y^2+2y+1\\ &\Rightarrow \quad x+3y=7\\ (x-2)^2+(y-5)^2 = (x+3)^2+(y-1)^2 \quad &\Rightarrow \quad x^2-4x+4+y^2-10y+25 = x^2+6x+9+y^2-2y+1\\ &\Rightarrow \quad 10x+8y=19\end{aligned}$$

Solving the systems of equations, we have $x = \frac{1}{22}$ and $y = \frac{51}{22}$. Our answer is $51+1+22 = 74$, **A.**

14. Call the smaller circle ω. Then the area of R can be found by subtracting the area of ω from the circular segment formed by $\overline{AB}$ and then dividing by two (since it is half of the remaining area). First, it is easy to find that since $\overline{AB}$ is at a distance of 2 from O, ω has a radius of $\frac{4-2}{2} = 1$ and thus has an area of π.

To find the area of the circular segment, we first find the area of sector AOB and then subtract the area of triangle AOB. It is easy to verify that $\angle AOB = 120°$. The area of the sector is then one-third the area of the circle, or $\frac{4^2\pi}{3} = \frac{16\pi}{3}$. It follows that $[\triangle AOB] = \frac{(4)(4)\sin 120°}{2} = 4\sqrt{3}$. Thus we have

$$[R] = \frac{1}{2}\left[\left(\frac{16\pi}{3} - 4\sqrt{3}\right) - \pi\right] = \frac{13\pi}{6} - 2\sqrt{3}, \ \textbf{B.}$$

15. Let the respective altitudes be a_1, a_2, and a_3. Then the area of the triangle can be written as $A = a_1 = 2a_2 = \frac{5a_3}{2}$. From this, we have $2a_1 = 4a_2 = 5a_3$. The least common multiple of 2, 4, and 5 is 20. Thus we can easily verify that the altitudes are in ratio $10 : 5 : 4$. We have $a+b+c = 10+5+4 = 19$, **C.**

16. The graph of $|x+y| < 2$ is the region between the lines $y = 2-x$ and $y = -2-x$. Thus the area we seek is just the area of this region for $0 < x < 2$. This is a parallelogram with base of length $2-(-2) = 4$ and height 2. The desired area is $(4)(2) = 8$, **D.**

17. Since $\overline{EF}$ is comprised of the two heights of the triangles, it has length equal to $2h = 2(2\sqrt{3}) = 4\sqrt{3}$. Furthermore the altitude of CEF to base $\overline{EF}$ is equal to half the side length of the square, or 2.

Thus we have $[CEF] = \frac{(4\sqrt{3})(2)}{2} = 4\sqrt{3}$, **C.**

18. Let the radius equal r and the height equal $3r$. Then we have $\frac{1}{3}\pi(r^2)(3r) = 1 \Rightarrow \pi r^3 = 1 \Rightarrow r = \dfrac{1}{\pi^{1/3}}$, **B.**

19. Call the points $(1,2)$, $(3,8)$, and $(4,1)$, A, B, and C, respectively. If $\overline{AB}$ is a side, then the fourth point is either $D = (4+2, 1+6) = (6,7)$ or $D = (4-2, 1-6) = (2,-5)$. If $\overline{BC}$ is a side, then the fourth point is either $D = (1+1, 2-7) = (2,-5)$ (which we already found), or $D = (1-1, 2+7) = (0,9)$. Therefore our answer is $6+7+0+9+2-5 = 19$, **B.**

    Remark. The three given points form the medial triangle of the triangle formed by the possible fourth points of the parallelogram.

20. It is clear that $CD = 1$ and $AC = \sqrt{2}$. Thus $\overline{CE}$ must equal 2 in order to retain the geometric progression. The height of $\triangle ABE$ is then $\sqrt{2^2 - (\frac{1}{2})^2} - 1 = \frac{\sqrt{15}}{2} - 1$. Thus the area is $(\frac{1}{2})(1)(\frac{\sqrt{15}}{2} - 1) = \frac{\sqrt{15}}{4} - \frac{1}{2}$, **A.**

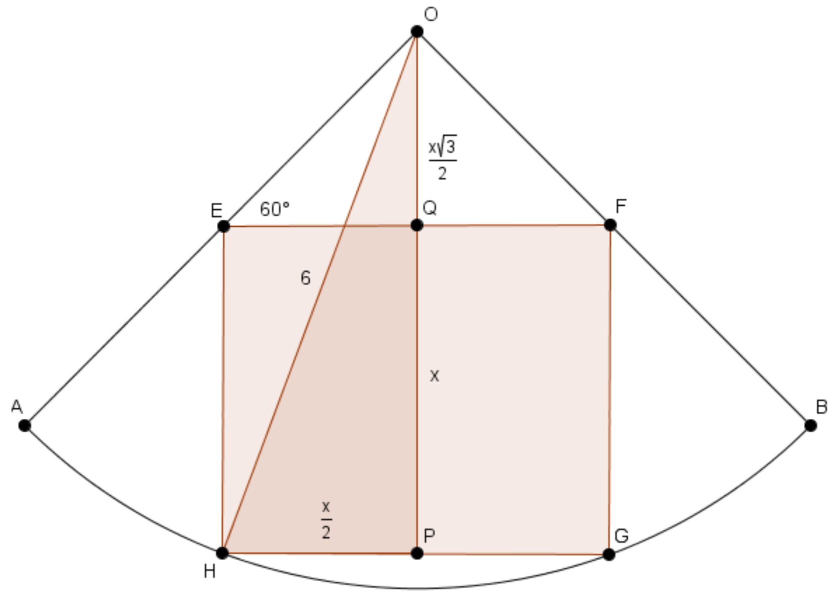
21. We have $V = \pi R^2 h = \pi(r^2)(\frac{1}{r^2}) = \pi$, **A.**

22. We can use Stewart's Theorem to compute the length of AM. Letting $AM = p$, we have

$$(4^2)(3) + (3^2)(3) = 6p^2 + (3)(3)(6) \Rightarrow p = \frac{\sqrt{14}}{2}.$$

    Now, we can apply Power of a Point around M. We have $(CM)(MB) = (AM)(MD) \Rightarrow MD = \frac{3^2}{\frac{\sqrt{14}}{2}} = \frac{9\sqrt{14}}{7}$, **A.**

23. Call the square $EFGH$ and call the sector OAB where O is the center, as shown below.



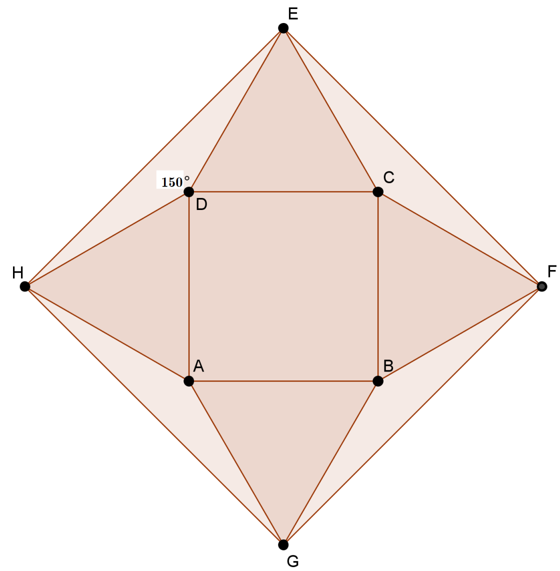
Let the side of the square be equal to x. If we draw segment $\overline{OH}$ and drop a perpendicular from O to $\overline{GH}$ at P, we obtain right triangle OPH. Since $\overline{OP}$ bisects $\angle AOB$, $\angle AOP = 30°$ and OEQ is a 30-60-90 triangle (where Q is the intersection of $\overline{OP}$ and $\overline{EF}$). Then clearly $OQ = \sqrt{3} \cdot EQ = \frac{x\sqrt{3}}{2}$. Hence $OP = x(\frac{\sqrt{3}}{2} + 1)$, $HP = \frac{x}{2}$, and $OH = 6$. Applying Pythagorean Theorem gives

$$\left(\frac{x}{2}\right)^2 + x^2\left(\frac{\sqrt{3}}{2}+1\right)^2 = 36 \Rightarrow \frac{x^2}{4} + x^2\left(\frac{7}{4}+\sqrt{3}\right) = 36$$
$$\Rightarrow x^2 = \frac{36}{2+\sqrt{3}}$$
$$\Rightarrow x^2 = 36(2-\sqrt{3})$$

Hence the area of the square is $36(2-\sqrt{3})$, **D.**

24. We have $A = rs$, or $8 \cdot \left(\frac{1}{2}\right) \cdot 18 = A = 72$, **B.**

25. The smallest square is formed by connecting the outer vertices of the equilateral triangles in consecutive order as shown in the following diagram.

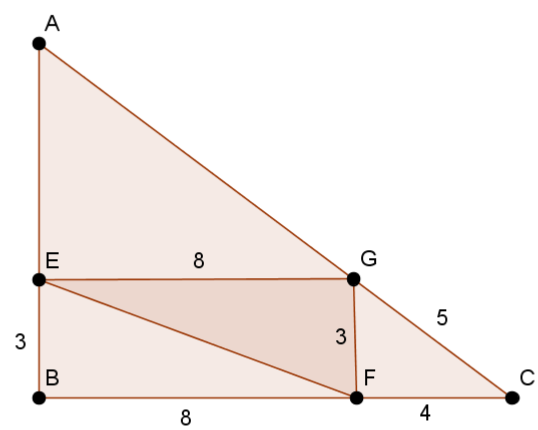


Let $EH = c$. Since $\angle EDC = \angle HDA = 60^\circ$ and $\angle CDA = 90^\circ$, $\angle EDH = 150^\circ$. We know that $ED = HD = DC = 4$, so applying Law of Cosines gives us

$$c^2 = 4^2 + 4^2 - (2)(4)(4)\cos 150^\circ = 32 + 16\sqrt{3},\ \textbf{D.}$$

26. It is well known that $F(n) = 360^\circ$. Thus we have $F(3) + F(4) + \cdots + F(n) = 360(n - 3 + 1) = 360n - 720$, **B.**

27. Consider the following diagram and notice that triangle GFC is a right triangle with sides of length 3, 4, and 5:



Similarly, it is easy to see that triangle AEG is a right triangle as well, with sides of length 6, 8, and 10. Thus we have $\overline{EG} || \overline{BC}$ and $\overline{GF} || \overline{AB}$ which implies that $\overline{EG} \perp \overline{GF}$. Hence triangle EGF is a right triangle with lengths of length 3 and 8. Its area is then $\frac{1}{2}(3)(8) = 12$, **C.**

28. Each successive square has exactly half the area of the previous one. Hence the sum is just an infinite geometric sequence with first term $10^2 = 100$ and common ratio $1/2$. Our answer is $\frac{100}{1-\frac{1}{2}} = 200$, **D.**

29. We simply have $AP^2 + CP^2 = BP^2 + DP^2 \Rightarrow 5^2 + 4^2 = 3^2 + DP^2 \Rightarrow DP = 4\sqrt{2}$, **C.**

30. Note that point A suffices the conditions for the British Flag Theorem on square $EBDF$. We have $FA^2 + BA^2 = CA^2 + EA^2 \Rightarrow EA^2 - FA^2 = 8^2 - 6^2 \Rightarrow (EA - FA)(EA + FA) = 28$, **D.**