ANSWERS

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SOLUTIONS

1. We have

\[
\sqrt{-1} \cdot \sqrt{-3} \cdot \sqrt{-11} \cdot \sqrt{-61} = i \cdot i \cdot i \cdot \sqrt{1 \cdot 3 \cdot 11 \cdot 61} = i^4 \sqrt{2013} = \sqrt{2013}, \text{ C.}
\]

2. The powers of \( i \) cycle between \( i, -1, -i, \) and 1. Thus, we have

\[
i^{2013} = (i^4)^{503} \cdot i = i, \text{ A.}
\]

3. The absolute value of the entire fraction is the ratio of the absolute values of the numerator and denominator. Using this, we have

\[
\left| \frac{3 + 4i}{5 + 12i} \right| = \left| \frac{3 + 4i}{5 + 12i} \right|
\]

\[
= \frac{5}{13}, \text{ A.}
\]
4. We let $x$ equal the expression we wish to evaluate. With a substitution, we obtain

$$x = \sqrt{\frac{i}{4}} + \sqrt{\frac{i}{4}} + \sqrt{\frac{i}{4}} + \cdots = \sqrt{\frac{i}{4}} + x.$$ Solving this equation with the quadratic formula gives

$$x^2 - x - \frac{i}{4} = 0 \Rightarrow x = \frac{1 \pm \sqrt{1+i}}{2}.$$

Now, we must evaluate $\sqrt{1+i}$. We can write this in cis form as $1+i = \sqrt{2}\cis\left(\frac{\pi}{4}\right)$. To take the square root of this, we utilize de Moivre's Theorem to obtain

$$\left[\sqrt{2}\cis\left(\frac{\pi}{4}\right)\right]^{1/2} = 2^{1/4}\cis\left(\frac{1}{2}\left(\frac{\pi}{4} + 2\pi k\right)\right), k = 0, 1
= 2^{1/4}\cis\left(\frac{\pi}{8} + \pi k\right), k = 0, 1.$$

Combining this with the rest of the solution gives

$$x = \frac{1 \pm 2^{1/4}\cis\left(\frac{\pi}{8} + \pi k\right)}{2}, k = 0, 1
= \frac{1}{2} + 2^{-3/4}\cis\left(\frac{\pi}{8} + \pi k\right), k = 0, 1, \quad [A]$$

5. Note that $\text{Re}(\cis\theta) = \cos\theta$ and $\text{Im}(\cis\theta) = \sin\theta$. Thus, we have

$$\prod_{n=1}^{45} \text{Re}[\cis((2n-1)\degree)] = \frac{\cos 1\degree \cos 3\degree \cdots \cos 89\degree}{\sin 2\degree \sin 6\degree \cdots \sin 178\degree}
= \frac{\cos 1\degree \cos 3\degree \cdots \cos 89\degree}{(2\sin 1\degree \cos 1\degree)(2\sin 3\degree \cos 3\degree)(2\sin 89\degree \cos 89\degree)}
= \frac{1}{2^{45}}\left(\frac{1}{\sin 1\degree \sin 3\degree \cdots \sin 89\degree}\right).$$

The bottom expression can be written as
\[
\sin 1^\circ \sin 3^\circ \ldots \sin 89^\circ = \frac{\sin 1^\circ \sin 2^\circ \sin 3^\circ \ldots \sin 89^\circ}{\sin 2^\circ \sin 4^\circ \ldots \sin 88^\circ} \\
= \frac{\sin 1^\circ \sin 2^\circ \sin 3^\circ \ldots \sin 89^\circ}{(2 \sin 1^\circ \cos 1^\circ)(2 \sin 2^\circ \cos 2^\circ) \ldots (2 \sin 44^\circ \cos 44^\circ)} \\
= \frac{1}{2^{45}} \left( \frac{\sin 45^\circ \sin 46^\circ \sin 47^\circ \ldots \sin 89^\circ}{\cos 1^\circ \cos 2^\circ \ldots \cos 44^\circ} \right) \\
= \frac{\sqrt{2}}{2^{45}} \left( \frac{\sin 46^\circ \sin 47^\circ \ldots \sin 89^\circ}{\sin 89^\circ \sin 88^\circ \ldots \sin 46^\circ} \right) \\
= 2^{-89/2},
\]

where we have used the fact that \( \sin(90^\circ - \theta) = \cos \theta \). Our answer is then

\[
2^{-45} \left( \frac{1}{2^{-89/2}} \right) = 2^{4/2}. \quad [B]
\]

6. The powers of \( i \) contain two sets of numbers that are additive inverses of each other, namely \((1, -1)\) and \((i, -i)\). Thus the only sets of four numbers that will satisfy \( a = 0 \) are permutations of either \((1, 1, -1, -1), \ (i, i, -i, -i), \) and \((i, -i, 1, -1)\). The first two have \( \binom{4}{2} = 6 \) distinct arrangements each, while the last has \( 4! = 24 \) total arrangements, giving \( 2(6) + 24 = 36 \) overall. There are \( 4^4 = 256 \) possibilities, giving a probability of

\[
\frac{36}{256} = \frac{9}{64}. \quad [B]
\]

7. The solutions to the equation \( z_k \) form a hexagon in the complex plane, similar to the 6\textsuperscript{th} roots of unity, except the side length of the hexagon is \( \sqrt{729} = 3 \). Thus \( |z_3 - z_6| \) is equal to the distance between two diagonally opposite points on the hexagon. This is simply \( 2(3) = 6 \). \quad [D]

8. We have \( v_1 = \langle a, b \rangle \) and \( v_2 = \langle c, d \rangle \), giving \( v_1 \cdot v_2 = ac + bd \). Intuition would lead us to try \( \text{Re}(z \cdot w) = ac - bd \). This, however, is the conjugate of what we wish to obtain. Naturally, we would then take the conjugate of either \( z \) or \( w \). This gives us

\[
\text{Re}(\overline{z} \cdot w) = \text{Re}((ac + bd) + i(ad - bc)) = ac + bd, \quad [D]
\]
9. Going by the definition, we have
\[
\left( \frac{i}{4} \right) = \frac{i(i-1)(i-2)(i-3)}{4!} = \frac{-10}{24} = -\frac{5}{12}, \ A.
\]

10. We have
\[
2(\text{cis}35^\circ \otimes \text{cis}35^\circ) = 2 \cos 35^\circ \text{cis}35^\circ
\]
\[
= 2 \cos 35^\circ (\cos 43^\circ + i \sin 43^\circ)
\]
\[
= 2 \cos 35^\circ \cos 43^\circ + i(2 \cos 35^\circ \sin 43^\circ)
\]
\[
= (\cos(43^\circ + 35^\circ) + \cos(43^\circ - 35^\circ)) + i(\sin(43^\circ + 35^\circ) + \sin(43^\circ - 35^\circ))
\]
\[
= (\cos 78^\circ + i \sin 78^\circ) + (\cos 8^\circ + i \sin 8^\circ)
\]
\[
= \text{cis}78^\circ + \text{cis}8^\circ.
\]

Thus, we have \[\theta \varphi = (78)(8) = 624, \ B.\]

11. Note that we can rewrite the equation as \((a-6)^2 + (b-3)^2 = 64\), or the equation for a circle. If we were to convert \(z\) to the Cartesian plane, we would simply write \(z = (x, y) = (a, b)\). Hence, \(R\) is a circle with radius 8, and thus has an area of \(8^2 = 64\pi = C.\)

12. Let \(z = a + bi\). Then we have
\[
|z - z| = |a + bi - a + bi| = |a + bi - \sqrt{a^2 + b^2}|
\]
\[
= |(a - \sqrt{a^2 + b^2}) + bi|
\]
\[
= \sqrt{a^2 - 2a\sqrt{a^2 + b^2} + (a^2 + b^2) + b^2}
\]
\[
= \sqrt{2(a^2 + b^2) - 2a\sqrt{a^2 + b^2}}.
\]

Now, since \(|z| = \sqrt{a^2 + b^2}\), this becomes
\[ \sqrt{2|z|^2 - 2a|z|} = \sqrt{2} \]
\[ \Rightarrow |z|^2 - a|z| - 1 = 0. \]

Using the quadratic formula, we solve for \(|z|\) as \(|z| = \frac{a \pm \sqrt{a^2 + 4}}{2} \). Since \(\sqrt{a^2 + 4} > 2 \), we can take \(|z| = \frac{a + \sqrt{a^2 + 4}}{2} \). \[ \text{D.} \]

13. We have

\[ v_1 \cdot v_2 = x (1 + i) + y (3 + 2i) \]
\[ = (x + 3y) + i (x + 2y) \]
\[ = 5 + 6i. \]

This gives us the systems of equations

\[ x + 3y = 5 \]
\[ x + 2y = 6 \]

which we solve as \((x, y) = (8, -1)\), which gives \(x + y = 7\), \[ \text{D.} \]

14. We have \( B = A - \lambda I = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 3 - \lambda & -2 \\ 4 & -1 - \lambda \end{pmatrix} \). Thus,

\[ \text{det}(B) = (3 - \lambda)(-1 - \lambda) - (-2)(4) \]
\[ = -3 - 3\lambda + \lambda + \lambda^2 + 8 \]
\[ = \lambda^2 - 2\lambda + 5 \]
\[ = 0. \]

Solving gives \( \lambda = \frac{2 \pm 4i}{2} = 1 \pm 2i \), \[ \text{A.} \]
15. It is clear that each \( f_n(x) \) will be a polynomial of degree 4, since the roots are the vertices of a square. Now, note that each set of roots is a rotation of \( \frac{\pi}{4} \) radians counterclockwise from the previous set of roots and furthermore, each set of roots has \( \frac{1}{\sqrt{2}} \) times the amplitude of the previous set of roots. We began with the fourth roots of unity, which are \( \text{cis} \left( \frac{\pi k}{2} \right) \), \( 0 \leq k \leq 3 \). This means the \( n \)th set of roots are 

\[
\left( \frac{1}{\sqrt{2}} \right)^{n-1} \text{cis} \left( \frac{\pi}{4} (n-1) + \frac{\pi k}{2} \right) = \left( \frac{1}{4} \right)^{n-1} \text{cis} \left( \pi(n-1) \right) .
\]

Of course, we can write this as 

\[
x^4 = \left( \frac{1}{4} \right)^{n-1} \text{cis} \left( \pi(n-1) \right) = (-1)^{n-1} \left( \frac{1}{4} \right)^{n-1} = \left( -\frac{1}{4} \right)^{n-1} .
\]

This implies that 

\[
f_n(x) = x^4 - \left( -\frac{1}{4} \right)^{n-1} .
\]

Thus we have 

\[
\sum_{n=1}^{\infty} f_n(0) = -\sum_{n=1}^{\infty} \left( -\frac{1}{4} \right)^{n-1} = -\frac{1}{1+\frac{1}{4}} = -\frac{4}{5}, \quad \text{[A]}
\]

16. The function will not intersect the \( x \)-axis when it has imaginary roots. This requires that the discriminant be less than 0. We have 

\[
5^2 - (4)(k^2)(9) < 0 \Rightarrow k^2 > \frac{25}{36} \Rightarrow k \in \left( -\infty, -\frac{5}{6} \right) \cup \left( \frac{5}{6}, \infty \right) , \quad \text{[D]}
\]

17. Let the roots be \( r_1, r_2, \ldots, r_{2013} \), where \( r_1 = 1 \). The sum of the roots taken two at a time can be written as \( \sum_{\text{cyc}} r_i r_j \), \( 0 < i, j \leq 2013, i \neq j \). This can be written as 

\[
\sum_{\text{cyc}} r_i r_j = \sum_{\text{cyc}} r_i r_a + \sum_{\text{cyc}} r_b r_c = \sum_{\text{cyc}} r_a + \sum_{\text{cyc}} r_b r_c ,
\]

Since \( r_1 = 1 \). We can see that this summation contains both the sum of the roots and the sum of the roots taken two at a time of \( g(x) = 1 + \sum_{n=4}^{2012} nx^n \). This is just 

\[
-\frac{2011}{2012} + \frac{2010}{2012} = -\frac{1}{2012} , \quad \text{[B]}
\]
18. We proceed by casework. Our first case, a real result, can be achieved by rolling both real numbers or both imaginary numbers. Note that both the first and second subcases are symmetric – so the total expected value is

\[
2 \left[ \frac{1}{36} \left( (4+5+6)(1+2+3) \right) \right] = 2 \cdot \frac{15 \cdot 6}{36} = 5.
\]

Our second case, an imaginary result, is achieved when we multiply an imaginary number by a real number. The expected value of this is

\[
-\frac{1}{36} \left[ (1+2+3)(1+2+3) + (4+5+6)(4+5+6) \right] = -\frac{29}{4}.
\]

The total expected value is \(5 - \frac{29}{4} = -2.25\). [A]

19. Writing the expression in \(\text{cis}\) form gives us

\[
(1+i\sqrt{3})^{2013} = \left[ 2 \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \right]^{2013}
\]

\[
= 2^{2013} \left( \text{cis} \left( \frac{\pi}{3} \right) \right)^{2013}
\]

\[
= 2^{2013} \text{cis} \left( \frac{2013\pi}{3} \right)
\]

\[
= 2^{2013} \text{cis} \pi
\]

\[
= -2^{2013}, [A].
\]

20. Let the first term be \(a\) and a common ratio be \(r\). If, at some point in the series, the \(n\)th term in the series equals the first, we have \(a = ar^{n-1} \Rightarrow r^n = 1, k = n-1\). Thus the possible ratios are the \(n\)th roots of unity. There must be 50 of these roots in the second quadrant, or between 90° and 180°. Since the roots of unity are \(\text{cis} \left( \frac{2\pi k}{n} \right)\), for some \(x\), we must have

\[
\frac{360x}{k} < 90 \Rightarrow x < \frac{k}{4}
\]

\[
\frac{360(x+50)}{k} < 180 \Rightarrow x < \frac{k}{2} - 50
\]
Subtracting the second from the first gives \( \frac{k}{4} > 50 \Rightarrow k > 200 \). Thus the smallest value of \( k \) is \( k = 201 \), which gives \( n = k + 1 = 201 + 1 = 202 \), [C].

21. This is just

\[
\begin{align*}
  f(i) &= 1 - \frac{i^2}{2!} + \frac{i^4}{4!} \\
       &= 1 + \frac{1}{2} + \frac{1}{24} \\
       &= \frac{37}{24}, [D].
\end{align*}
\]

22. Note that \( \text{cis} \theta_1 \text{cis} \theta_2 = \text{cis}(\theta_1 + \theta_2) \). Using this, we have

\[
\prod_{\theta=1}^{2013} \text{cis} \theta^\circ = \text{cis} 1^\circ \text{cis} 2^\circ \cdots \text{cis} 2013^\circ \\
= \text{cis} \left( \frac{2013(2014)}{2} \right) \\
= \text{cis} (1007 \cdot 2013)^\circ, [B].
\]

23. The plotted points form a spiral shape, composed of segments which we can treat as hypotenuses of right triangles for our purposes of calculating distance. Since the powers of \( i \) traverse the axes counterclockwise, each two set of consecutive points along with the origin form a right triangle. For example, \( z_1 = \sqrt{\binom{2}{2}} = i \), and

\[
\begin{align*}
  z_2 &= \sqrt{\binom{3}{2}} = -\sqrt{3}, \text{ giving } z_1 z_2 = \sqrt{1^2 + (\sqrt{3})^2} = 2. \text{ In general, we have }
  \\
  z_k z_{k+1} &= \sqrt{\frac{k+1}{2} + \frac{k+2}{2}} \\
           &= \sqrt{\frac{k(k+1)}{2} + \frac{(k+1)(k+2)}{2}} \\
           &= \sqrt{(k+1)^2} \\
           &= k+1.
\end{align*}
\]
Thus,
\[ z_1z_2 + z_2z_3 + \cdots + z_{2012}z_{2013} = 2 + 3 + \cdots + 2013 \]
\[ = \frac{2013 \cdot 2014}{2} - 1 \]
\[ = 2027090. \]

Therefore, our answer is 2027090(mod100) ≡ 90, [C].

24. Note that \( A \) is a 60° counterclockwise rotation matrix. So every \( \frac{360°}{60°} = 6 \) times we apply it, we simply return to the same vector. This means that

\[ A^{37}z = A^{6(6)+1}z = Az \]
\[ = \begin{pmatrix} 1 & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 3 \\ 4i \end{pmatrix} \]
\[ = \frac{1}{2} \begin{pmatrix} 3 - 4i\sqrt{3} \\ 3\sqrt{3} + 4i \end{pmatrix}, \quad [B]. \]

25. We have
\[ \sum_{n=1}^{k} n(n!) = \sum_{n=1}^{k} (n+1-1)(n!) \]
\[ = \sum_{n=1}^{k} [(n+1)(n!) - n!] \]
\[ = \sum_{n=1}^{k} [(n+1)! - n!] \]
\[ = [(k+1)!(k!)+[k!+(k-1)!]+\cdots+[2!-1!]] \]
\[ = (k+1)! - 1 \]

Thus, the sum becomes
\[ \sum_{n=1}^{k} n \cdot n! + 1 = (k+1)! - 1 + 1 = (k+1)!, \]
and we have
\[ \sum_{k=0}^{\infty} \frac{i^k}{(k+1)!} = \frac{1}{i} \sum_{k=0}^{\infty} \frac{i^{k+1}}{(k+1)!} \]
\[ = \frac{1}{i} \left( \sum_{n=1}^{\infty} \frac{i^n}{n!} \right) \]
\[ = -i(e^i - 1) \]
\[ = -ie^i + i, \text{ C.} \]

26. We can write \( f(x) \) as
\[ f(x) = x^{2013} + x^{2012} + \cdots + x + 1 \]
\[ = x^{2012}(x+1) + x^{2010}(x+1) + \cdots + x^2(x+1) + (x+1) \]
\[ = (x+1)(x^{2012} + x^{2010} + \cdots + x^2 + 1) \]
\[ = (x+1)(x^{2010}(x^2 + 1) + \cdots + (x^2 + 1)) \]
\[ = (x+1)(x^2 + 1)(x^{2010} + x^{2008} + \cdots + 1). \]

Thus, we know that \( f(x) \) has \(-1, i\), and \(-i\) as roots. Since the powers of \( i \) cycle, we are only worried about the powers of \( i \) that come out to 1, or every fourth power. Note that \( R(1) = f(1) = 2014 \), by the Remainder theorem. Since we begin at \( k = 0 \), our answer is
\[ 2014 \left( \frac{2013}{4} + 1 \right) \equiv 1015056 \equiv 56, \text{ D.} \]

27. The function in this problem is similar to the function given in Problem 26. We can write \( f_n(x) \) as
\[ f_n(x) = \sum_{j=0}^{2^n-1} x^j \]
\[ = \sum_{j=0}^{2^n-1} \left( x^{2^j} + x^{2^{j+1}} \right) \]
\[ = (1 + x) \sum_{j=0}^{2^n-1} x^{2^j} \]
\[ = (1 + x) \sum_{j=0}^{2^n-1} \left( x^{4^j} + x^{4^{j+1}} \right) \]
\[ = (1 + x)(1 + x^2) \sum_{j=0}^{2^n-1} x^{4^j} \]
\[ \vdots \]
\[ = (1 + x)(1 + x^2) \cdots \left( 1 + x^{2^{n-1}} \right). \]
Solving $x^{2^{n-1}} = -1$ gives us $x = \text{cis}\left(\frac{\pi}{2^{n-1}} + \frac{\pi k}{2^{n-2}}\right)$, where $\psi_n = \left\{\frac{\pi}{2^{n-1}}, \frac{3\pi}{2^{n-1}}, \ldots, \frac{(2^{n-1}-1)\pi}{2^{n-1}}\right\}$. Note that the entire set $\psi$ is a union of all $\psi_i$. The sum for a given $\psi_n$ is

$$\frac{\pi}{2^{n-1}} \left(1 + 3 + \cdots + (2^{n-1} - 1)\right) = \frac{\pi}{2^{n-1}} \left(2^{n-2}\right)^2 = \frac{2^{2n-4}}{2^{n-1}} \pi = 2^n \left(\frac{\pi}{8}\right).$$

Thus the entire sum (while accommodating for $x+1 = 0 \Rightarrow x = \text{cis}(\pi)$) is

$$\pi + \frac{\pi}{8} \left(2^1 + 2^2 + \cdots + 2^n\right) = \pi + \frac{\pi}{8} \left(\frac{2^n - 1}{1}\right) = \pi + \frac{\pi}{4} \left(2^n - 1\right).$$

Finally, we must find $n$ such that

$$\pi + \frac{\pi}{4} \left(2^n - 1\right) > 2013 \pi \Rightarrow 2^n - 1 > 8048 \Rightarrow n > \log_2 8049.$$

We can easily verify that the smallest such $n$ is 12, [C].

28. We know that $|z|^2 = m^2 + 9n^2$. Consider this modulo 8. Since the quadratic residues mod 8 are 0, 1, and 4, the possible values of $|z|^2 \pmod{8}$ are

$$\begin{align*}
0 + 0 & \equiv 0 \pmod{8} \\
0 + 1 & \equiv 1 \pmod{8} \\
0 + 4 & \equiv 4 \pmod{8} \\
1 + 1 & \equiv 2 \pmod{8} \\
1 + 4 & \equiv 5 \pmod{8}
\end{align*}$$

The answer choices, mod 8, are
2010 \equiv 2 \pmod{8} \\
2011 \equiv 3 \pmod{8} \\
2012 \equiv 4 \pmod{8} \\
2013 \equiv 5 \pmod{8} \\

Thus our answer is 2011, B.

29. Our intuition for a new set of axes is based on the fact that the Eisenstein integers have an argument of 60°. Through some playing around, we can find the set of axes as shown below:

As we can see, the plotted points form an equilateral triangle with a side length of 3. Thus, the area is \( \frac{\sqrt{3}}{4} \cdot 9 \sqrt{3} = \frac{9 \sqrt{3}}{4} \), C.

30. We have

\[
z = a + b\omega = a + b \left( \frac{1}{2}(-1 + i \sqrt{3}) \right) = \left( a - \frac{b}{2} \right) + i \left( \frac{b \sqrt{3}}{2} \right).
\]

Thus,
\[ |z| = \sqrt{\left( a - \frac{b}{2} \right)^2 + \left( \frac{b\sqrt{3}}{2} \right)^2} \]

\[ = \sqrt{a^2 - ab + \frac{b^2}{4} + \frac{3b^2}{4}} \]

\[ = \sqrt{a^2 - ab + b^2} \]

\[ = \sqrt{(a - b)^2 + ab}, \quad \text{(C)} \]