Answers:

1. D
2. A
3. C
4. D
5. C
6. B
7. A
8. C
9. D
10. B
11. B
12. A
13. D
14. C
15. A
16. C
17. B
18. A
19. A
20. C
21. E
22. B
23. D
24. C
25. A
26. D
27. B
28. C
29. D
30. B

TB1. 3570
TB2. 35
TB2. 492

Solutions:

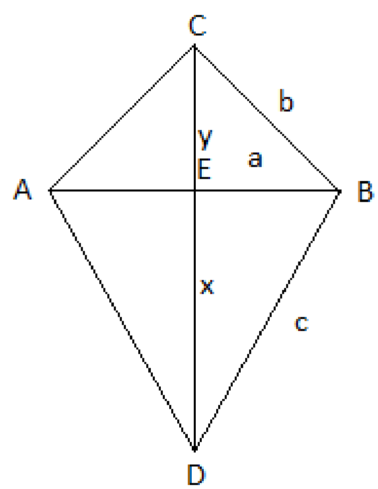
1. $\left\|\vec{v}\right\| = \left\|\langle -28,12,21\rangle\right\| = \sqrt{(-28)^2 + 12^2 + 21^2} = \sqrt{784+144+441} = \sqrt{1369} = 37$

2. $\left\lfloor \sum_{n=0}^{5} \sin(15n)^\circ \right\rfloor = \lfloor \sin 0^\circ + \sin 15^\circ + \sin 30^\circ + \sin 45^\circ + \sin 60^\circ + \sin 75^\circ \rfloor$

$= \left\lfloor 0 + \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{6}+\sqrt{2}}{4} \right\rfloor = \left\lfloor \frac{1+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2} \right\rfloor = \lfloor 3.2978 \rfloor = 3$

3. $\tan\theta = \sqrt{3}$ corresponds to $\theta = 60^\circ$ or $\theta = 240^\circ$ in the first rotation around the circle. $\theta = 60^\circ$ corresponds to $r^2 = 4\cos(3\theta) = -4$, which has no solutions. $\theta = 240^\circ$ corresponds to $r^2 = 4\cos(3\theta) = 4$, so $r = \pm 2$, giving two points of intersection. However, both graphs go through the origin as well, so there are a total of three points of intersection.

4. Label the edges of the kite as in the picture. Therefore, we have $a^2 + x^2 = c^2$ and $a^2 + y^2 = b^2$. Subtract the second equation from the first to get $x^2 - y^2 = c^2 - b^2$, which equals 25 by assumption. Since $x + y = 25$ also, we must have $x - y = 1$ since $25 = x^2 - y^2 = (x+y)(x-y)$. Solving this system yields $x = 13$, which is the sought length.

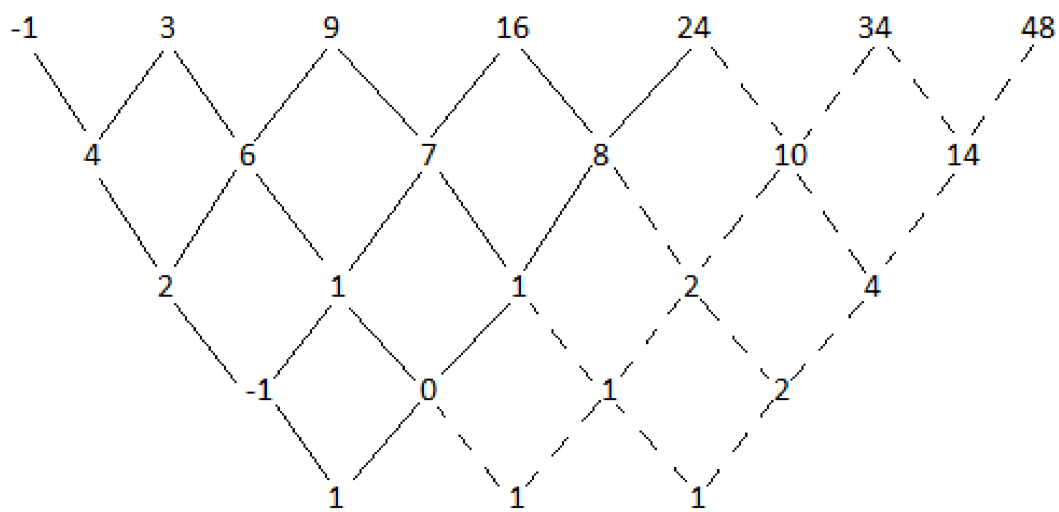


5. First, we will find all ordered triples (a,b,c) with nonnegative coordinates with $a < b < c$. If $a = 0$, then $b^2 + c^2 = 225$, and the only solutions are $(0,0,15)$ and $(0,9,12)$. A quick search yields no solutions for $a = 1$ (the same is the case for $a = 3,\ 4,\ 6,\ 7,$ or 8; it isn't necessary to check $a = 9$ or larger values since in those cases we would have $a^2 + b^2 + c^2 \geq 3\cdot 9^2 = 243 > 225$). If $a = 2$, then $b^2 + c^2 = 221$, and the only solutions are $(2,5,14)$ and $(2,10,11)$. If $a = 5$, then $b^2 + c^2 = 200$, and the only solution is $(5,10,10)$. For $(2,5,14)$ and $(2,10,11)$, there are six different arrangements, and each number could be positive or negative, so for the two points, there are $2\cdot 6\cdot 2^3 = 96$ points. For $(5,10,10)$, there are three different arrangements, and each number could be positive or negative, so for that point there are $3\cdot 2^3 = 24$ points. For $(0,9,12)$, there are six different arrangements, and each nonzero number could be positive or negative, so for that point there are $6\cdot 2^2 = 24$ points. Finally, for $(0,0,15)$, there are three

different arrangements, and each nonzero number could be positive or negative, so for that point there are $3 \cdot 2 = 6$ points. This gives a total of $96 + 24 + 24 + 6 = 150$ distinct points.

6. Each factor of the equation corresponds to a single point, and these can be found by completing the square. $x^2 + y^2 - 2x + 4y + 5 = (x-1)^2 + (y+2)^2$, so if this equals 0, the point is $(1,-2)$. $x^2 + y^2 + 4x + 2y + 5 = (x+2)^2 + (y+1)^2$, so if this equals 0, the point is $(-2,-1)$. The distance between these points is $\sqrt{(1+2)^2 + (-2+1)^2} = \sqrt{9+1} = \sqrt{10}$.

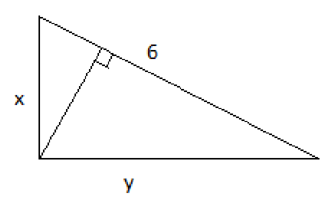
7. Using the method of finite differences, we have:



The given information is connected with solid lines, so the minimum degree of f is 4. Finishing out the rest of the differences (dashed lines) to get two more values, working backwards from 1's along the bottom row, yields $f(7) = 48$.

8. Since $\cos 53^\circ = \sin 37^\circ$, the angles being complementary, we must solve $\cos 37^\circ = \sin\theta$. Again, the complementary angle would be the smallest positive angle that works, so the answer is $\theta = 53^\circ$.

9. In the picture, based on the given information, $x = \sqrt{13} - \sqrt{5}$, and using the Pythagorean Theorem, $y = \sqrt{13} + \sqrt{5}$. This makes the altitude to the hypotenuse $\dfrac{\left(\sqrt{13}-\sqrt{5}\right)\left(\sqrt{13}+\sqrt{5}\right)}{6} = \dfrac{8}{6} = \dfrac{4}{3}$.



10. $0=\begin{vmatrix}1&2&3\\x&5&0\\4&x&6\end{vmatrix}=30+0+3x^2-60-0-12x=3x^2-12x-30$, so the sum of the solutions is $-\frac{-12}{3}=4$ since the two solutions are different.

11. $f\left(f\left(c\right)\right)=f\left(\frac{2c-1}{1+2c}\right)=\frac{2\left(\frac{2c-1}{1+2c}\right)-1}{1+2\left(\frac{2c-1}{1+2c}\right)}=\frac{2c-3}{6c-1}$ and $f\left(c-1\right)=\frac{2\left(c-1\right)-1}{1+2\left(c-1\right)}=\frac{2c-3}{2c-1}$, so setting these equal mean that $2c-3=0\Rightarrow c=\frac{3}{2}$ or $6c-1=2c-1\Rightarrow c=0$. Therefore, the nonzero solution is $c=1.5$.

12. Reordering the 12-element data set, we get 1, 1, 2, 3, 3, 5, 5, 7, 7, 7, 7, 12, and the sum of the elements is 60, so $A=5$. The median would be the average of 5 and 5, so $B=5$. Clearly, $C=7$, so $C-A-B=7-5-5=-3$.

13. The side length of the hexagon is $2\cdot\frac{4}{\sqrt{3}}=\frac{8}{\sqrt{3}}$, so the enclosed area is $\frac{3\left(\frac{8}{\sqrt{3}}\right)^2\sqrt{3}}{2}=32\sqrt{3}$.

14. Let $x=-\sqrt{5-\sqrt{5-\sqrt{5-\sqrt{5-\ldots}}}}$. Then $x=-\sqrt{5+x}$ and $x<0$. Thus $x^2=5+x\Rightarrow$ $0=x^2-x-5\Rightarrow x=\frac{1\pm\sqrt{21}}{2}$. Since $x<0$, $x=\frac{1-\sqrt{21}}{2}$.

15. Counting up the white area as fractions of the total, we get $\frac{1}{4}+\frac{1}{4}\left(\frac{3}{4}\right)+\frac{1}{4}\left(\frac{3}{4}\right)^2+\frac{1}{4}\left(\frac{3}{4}\right)^3+\ldots$ which is an infinite geometric series with sum $\frac{\frac{1}{4}}{1-\frac{3}{4}}=1$. So, actually, none of the triangle's area is black!

16. $\prod_{n=1}^{18}\left(2^n\sin\left(20n-10\right)^\circ\right)=2^{\frac{18\cdot19}{2}}\prod_{n=1}^{18}\left(\sin\left(20n-10\right)^\circ\right)$, and using the fact that

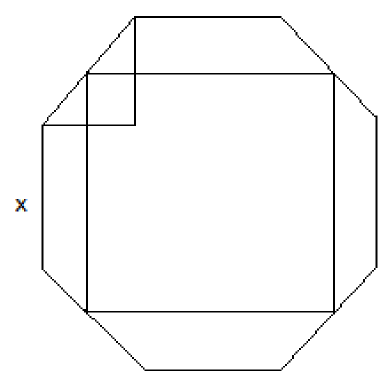
$\sin 10^\circ \sin 50^\circ \sin 70^\circ = \frac{1}{8}$, we get $2^{\frac{18\cdot 19}{2}} \prod_{n=1}^{18} \left(\sin(20n-10)^\circ \right) = 2^{171} \left(\frac{1}{8}\right)^2 \left(\frac{1}{2}\right)^2 (1) \left(-\frac{1}{8}\right)^2 \left(-\frac{1}{2}\right)^2 (-1)$
$= -\frac{2^{171}}{2^{16}} = -2^{155}$.

17. x must be coterminal with $\frac{\pi}{6}$ or $\frac{5\pi}{6}$. Since $\frac{241\pi}{12} = 20\pi + \frac{\pi}{12}$, the largest coterminal angle is $18\pi + \frac{5\pi}{6} = \frac{113\pi}{6}$.

18. The five least nonnegative even integers are 0, 2, 4, 6, and 8. The four greatest odd negative integers are -7, -5, -3, and -1. The sum of these numbers is $(0+2+4+6+8)-(1+3+5+7) = 20-16 = 4$.

19. The sum of the reciprocals of the positive integral factors is equal to the sum of the positive integral factors divided by the number itself. Since 28 is a perfect number, the sum of the positive integral factors is twice 28, and the sought sum is $\frac{56}{28} = 2$ (it doesn't take much to verify this by writing out the positive integral factors of 28, and it turns out that the sum of the reciprocals of the positive integral factors of any perfect numbers is always 2).

20. Let x be the length of the side of the octagon, as in the diagram. Considering the octagon as a square with four isosceles right triangles removed, the enclosed area of the octagon is $\left(x + 2\cdot\frac{x}{\sqrt{2}} \right)^2 - 4\cdot\frac{1}{2}\left(\frac{x}{\sqrt{2}}\right)^2 = x^2\left(2+2\sqrt{2}\right)$. The square encloses an area of $\left(x + 2\cdot\frac{x}{2\sqrt{2}} \right)^2 = x^2\left(\frac{3}{2}+\sqrt{2}\right)$, so the sought ratio is $\frac{\frac{3}{2}+\sqrt{2}}{2+2\sqrt{2}} = \frac{1+\sqrt{2}}{4}$.



21. $(\csc\theta - \cot\theta)(\csc\theta + \cot\theta) = \csc^2\theta - \cot^2\theta = 1$, which is none of the answer choices.

22. Let $S = \sum_{n=1}^{\infty} \left((n^2+2n-4)\left(\frac{2}{3}\right)^n \right) = -\left(\frac{2}{3}\right) + 4\left(\frac{2}{3}\right)^2 + 11\left(\frac{2}{3}\right)^3 + 20\left(\frac{2}{3}\right)^4 + \ldots$. Multiply this

equation by $\frac{2}{3}$ to get $\frac{2}{3}S=-\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)^3+11\left(\frac{2}{3}\right)^4+\ldots$. Now, subtract this second equation from the first equation to get $\frac{1}{3}S=-\left(\frac{2}{3}\right)+5\left(\frac{2}{3}\right)^2+7\left(\frac{2}{3}\right)^3+9\left(\frac{2}{3}\right)^4+\ldots$. Now multiply this equation by $\frac{2}{3}$ to get $\frac{2}{9}S=-\left(\frac{2}{3}\right)^2+5\left(\frac{2}{3}\right)^3+7\left(\frac{2}{3}\right)^4+\ldots$, and subtract that from the previous equation to get $\frac{1}{9}S=-\left(\frac{2}{3}\right)+6\left(\frac{2}{3}\right)^2+2\left(\frac{2}{3}\right)^3+2\left(\frac{2}{3}\right)^4+\ldots=-\frac{2}{3}+\frac{8}{3}+\frac{\frac{16}{27}}{1-\frac{2}{3}}=2+\frac{16}{27}\cdot 3=\frac{34}{9}$.

Therefore, $S=34$.

23. $x=\cos\left(\frac{4\pi}{3}-\cos^{-1}x\right)=-\frac{1}{2}x+\left(-\frac{\sqrt{3}}{2}\right)\sqrt{1-x^2}\Rightarrow\frac{3}{2}x=-\frac{\sqrt{3-3x^2}}{2}$. Squaring both sides yields $9x^2=3-3x^2\Rightarrow 12x^2=3\Rightarrow x=\pm\frac{1}{2}$. However, the positive solution does not work because it gives a positive number on the left side of the equation just before squaring and a negative number on the right side. The negative number does work, so $x=-\frac{1}{2}$ is the only solution.

24. $(\sin x)(\sin(3x))+(\cos x)(\cos(3x))=\cos(3x-x)=\cos(2x)$ (even if you chose to go with $\cos(x-3x)=\cos(-2x)$, the cosine function is even and thus yields $\cos(2x)$ also).

25. For $(x-4)^2-(y+3)^2=1$, using the standard conic section notation, $a=b=1$, and since the latus rectum has length $\frac{2b^2}{a}$, the sought length is $\frac{2(1)^2}{1}=2$.

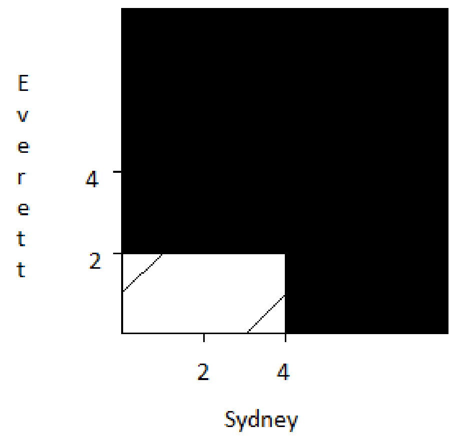
26. $\frac{\left(1-\sqrt{3}i\right)^{24}}{\left(\sqrt{2}+\sqrt{2}i\right)^m}=\frac{\left(2cis300^\circ\right)^{24}}{\left(2cis45^\circ\right)^m}=\frac{2^{24}cis7200^\circ}{2^m cis\left(45^\circ m\right)}=\frac{2^{24-m}}{cis\left(45^\circ m\right)}$. To make this an integer, we must have $0<m\le 24$ and m must be a multiple of 4 so that $45^\circ m$ is coterminal with either 0° or 180°. Thus, the valid values of m are 4, 8, 12, 16, 20, and 24, and their sum is $4+8+12+16+20+24=84$.

27. Let x_n be the number of ways to get to the nth step. Clearly $x_1=0$ and $x_2=x_3=1$, but

also $x_n = x_{x-2} + x_{n-3}$, so the sequence is 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, and the 17th term is 49.

28. Let s_n be the sum of the nth powers of the solutions of the equation. Therefore, $s_1 = -\frac{0}{2} = 0$. Using the method of Newton's sums, we have $0 = 2s_2 + 0s_1 - 3\cdot 2 \Rightarrow s_2 = 3$, and finally, $0 = 2s_3 + 0s_2 - 3s_1 - 3\cdot 3 \Rightarrow s_3 = 4.5$.

29. Imagine the 8×8 grid in the plane, where the x-coordinate represents the time in hours after 10 pm when Sydney arrives and the y-coordinate represents the time in hours after 10 pm when Everett arrives. By the given information about the latest each could arrive at the rave, we reduce the possible area down to the 4×2 rectangle. If Everett arrives at 12 am, the latest he could arrive, Sydney would have to arrive no earlier than 11 pm in order to have 3 hours of overlap (this point is represented by the point $(1,2)$). If Sydney arrives at

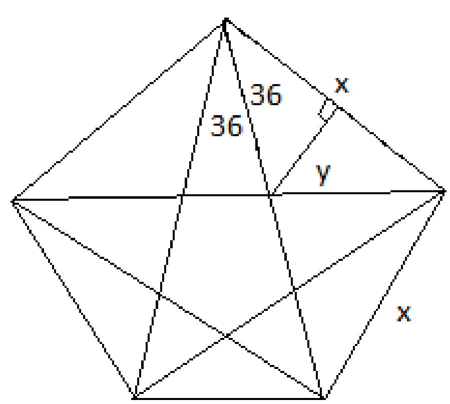


2 am, the latest she could arrive, Everett would have to arrive no earlier than 11 pm in order to have 3 hours of overlap (this point is represented by the point $(4,1)$). From those points, draw line segments with a slope of 1 to the edge since if each back their arrival time up the same amount from those times, they would still overlap for at least 3 hours. The favorable area is between the line segments, and that area is $4\cdot 2 - 2\cdot\frac{1}{2}\cdot 1\cdot 1 = 7$, and the total possible area is 8, so the probability of their overlap lasting at least 3 hours is $\frac{7}{8}$.

30. Let d be the length of a diagonal of the larger pentagon, and let x be the length of a side of the larger pentagon. Then, by the Law of Cosines,

$$d^2 = 2x^2 - 2x^2\cos 108^\circ = 2x^2\left(1-\cos 108^\circ\right)$$

$$= 4x^2\left(\frac{1-\cos 108^\circ}{2}\right) = 4x^2\sin^2 54^\circ \Rightarrow d = 2x\sin 54^\circ.$$



Let y be the length of the part of a diagonal from a vertex of the smaller pentagon to one of its closest vertices of the larger pentagon, and drop an altitude from the vertex of the smaller pentagon to one of the sides of the larger pentagon closest to it.

Then $\cos 36^\circ = \frac{x/2}{y} = \frac{x}{2y} \Rightarrow 2y = \frac{x}{\cos 36^\circ} = \frac{x}{\sin 54^\circ}$. Therefore, the side length of the smaller pentagon would be $2x\sin 54^\circ - \frac{x}{\sin 54^\circ} = x\left(\frac{2\sin^2 54^\circ - 1}{\sin 54^\circ}\right) = x\left(\frac{2\cos^2 36^\circ - 1}{\sin 54^\circ}\right) = x\left(\frac{\cos 72^\circ}{\sin 54^\circ}\right)$

$= x\left(\frac{\cos 72^\circ}{\cos 36^\circ}\right)$, making the ratio of the smaller pentagon's side length to the larger pentagon's side length $\frac{x\left(\frac{\cos 72^\circ}{\cos 36^\circ}\right)}{x} = \frac{\cos 72^\circ}{\cos 36^\circ}$. Since the larger pentagon's side length is $\cos 36^\circ$, the length of the side of the smaller pentagon is $\cos 72^\circ$.

Tiebreakers

TB1. $\frac{84 \cdot 85}{2} = 3570$

TB2. $3570 = 85 + 86 + ... + (84+n) = \frac{n}{2}\left(85 + (84+n)\right) \Rightarrow 7140 = n^2 + 169n \Rightarrow 0 = n^2 + 169n - 7140$
$= (n-35)(n+204) \Rightarrow n = 35$ (since $n > 0$).

TB3. $1 + 2 + ... + n = (n+1) + (n+2) + ... + (n+m) \Rightarrow \frac{n(n+1)}{2} = \frac{m}{2}\left((n+1) + (n+m)\right)$

$n^2 + n = 2nm + m + m^2 \Rightarrow 0 = m^2 + (2n+1)m - (n^2+n) \Rightarrow m = \frac{-(2n+1) \pm \sqrt{4n^2 + 4n + 1 + 4(n^2+n)}}{2 \cdot 1}$

$\Rightarrow m = \frac{-2n - 1 + \sqrt{8n^2 + 8n + 1}}{2}$ $(\text{since } m > 0)$, so we are looking for the least $n > 84$ such that $8n^2 + 8n + 1 = 2(2n+1)^2 - 1$ is a perfect square. Setting this perfect square as k^2 and setting $p = 2n+1$, we are trying to solve the Pell Equation $k^2 - 2p^2 = -1$. In trying to solve $k^2 - 2p^2 = \pm 1$, find the smallest solution in positive integers, which is $k = p = 1$. All solutions are found recursively in the following way:

1. The next p is the sum of the previous k and p.
2. The next k is the sum of the previous k and twice the previous p.

The table looks like this:

k	p	k^2-2p^2	n (only when p is odd)
1	1	-1	0
3	2	1	
7	5	-1	2
17	12	1	
41	29	-1	14
99	70	1	
239	169	-1	84
577	408	1	
1393	985	-1	492

It can be shown also that the 1 and -1 will always alternate, so the next value of n will be 492. In fact, were the last column to be continued, the numbers in the last column would be the only numbers that have the property defined in this question.