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<th>Question</th>
<th>Solution</th>
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<td><strong>P1.</strong></td>
<td>The common difference of an arithmetic sequence is the difference between a term and the previous term (in that order); in this case, $23 - (-10) = 33$.</td>
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<td><strong>P2.</strong></td>
<td>Since $2^{13} = 32768$, the digital sum is $3 + 2 + 7 + 6 + 8 = 26$.</td>
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<td><strong>P3.</strong></td>
<td>The smaller value is $k = -4$.</td>
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<td><strong>P4.</strong></td>
<td>We have $\csc(330^\circ) = -2$.</td>
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<td><strong>P5.</strong></td>
<td>We have $(A - B)^{D-C} = (33 - 26)^{-2-(-4)} = 7^2 = 49$.</td>
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<tr>
<td><strong>1.</strong></td>
<td>If $x = 3$, then $</td>
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<td><strong>2.</strong></td>
<td>The period of $y = \sin\left(\frac{\pi x}{6}\right)$ is $\frac{2\pi}{\pi/6} = 12$. The absolute value cuts the period in half, since portions below the $x$-axis get reflected to positive values, so the graph starts the cycle quicker. The answer is $6$.</td>
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<td><strong>3.</strong></td>
<td>We have $8\sqrt{2}\cos\left(\frac{7\pi}{4}\right) - 6\sqrt{3}\sin\left(\frac{4\pi}{3}\right) = 8\sqrt{2}\left(\frac{1}{\sqrt{2}}\right) - 6\sqrt{3}\left(\frac{-\sqrt{3}}{2}\right) = 8 + 9 = 17$.</td>
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<td><strong>4.</strong></td>
<td>We have $4^x - 4^{x-1} = 4^x - \left(\frac{1}{4}\right)4^x = \left(\frac{3}{4}\right)4^x = 24$, so $4^x = 2^{2x} = 32 = 2^5$, so $2x = 5$. Therefore, $(2x)^{2x} = 5^5 = 3125$.</td>
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<td><strong>5.</strong></td>
<td>We have $AB - C + D = (3)(6) - 17 + 3125 = 3126$.</td>
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<td><strong>6.</strong></td>
<td>Since $2^{12} &lt; 5566 &lt; 2^{13}$, the binary representation of 5566 is a 1 followed by 12 other digits for a total of 13 digits.</td>
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7. For convenience’s sake, let square $ABCD$ have a side length of 2. We know that triangles $AED$ and $DFC$ are congruent. Let $\phi = \angle EDA = \angle FDC$ so that $\theta = \frac{\pi}{2} - 2\phi$; thus, $\sin \theta = \sin \left(\frac{\pi}{2} - 2\phi\right) = \cos(2\phi) = 1 - 2 \sin^2 \phi$. Since $\sin \phi = \frac{1}{\sqrt{5}}$, we have $\sin \theta = 1 - 2 \sin^2 \phi = 1 - 2 \left(\frac{1}{\sqrt{5}}\right)^2 = 1 - \frac{2}{5} = \frac{3}{5}$.

8. If $\cos x = \frac{\sqrt{3}}{5}$, then $\cos^2 x = \frac{3}{25}$ and $\sin^2 x = 1 - \frac{3}{25} = \frac{22}{25}$. Consequently, $\cot^2 x = \frac{3/25}{22/25} = \frac{3}{22}$, making the answer $484 \left(\frac{3}{22}\right) = 66$.

9. The relation can be expressed as $f = Kg^2h$ for some $K$, which, in this case, is equal to $K = \frac{f}{g^2h} = \frac{128}{4^2 \times 2} = 4$. The answer is then $f = 4(3^2)(6) = 216$.

10. We have $AB(D - C) = (13) \left(\frac{2}{5}\right)(216 - 66) = 1170$.

11. If $L(x) = mx + b$, then $I(x) = \frac{x-b}{m}$. So we have $mx + b = \frac{4(x-b)}{m} + 3$, or $mx + b = \frac{4x}{m} + 3 - \frac{4b}{m}$. Set corresponding coefficients equal to each other to obtain the equations $m = 4/m$ and $b = 3 - \frac{4b}{m}$. Since the slope is positive, $m = 2$. Plug this into the second equation to get $b = 3 - \frac{4b}{2} = 3 - 2b$, or $b = 1$. Thus, $L(10) = 2(10) + 1 = 21$.

12. By the Power-Reducing formula $\cos^2 x = \frac{1 + \cos(2x)}{2}$, we have $\cos^2 x = \frac{1 + \frac{3}{7}}{2} = \frac{5}{7}$ so that $m + n = 12$.

13. Since $\sin \frac{11\pi}{6} = -\frac{1}{2}$ we have $\arcsin(-\frac{1}{2}) = -\frac{\pi}{6}$.

14. For positive $x$, $g^{-1}(x) = x^2 - 2$ (this is easy to check just by plugging it into $g$ and seeing that the identity function is obtained), so $g^{-1}(5) = 23$ and $f(g^{-1}(5)) =$
f(23) = 65.

15. We have $AB \cot^2 C + 10D = (21)(12) \cot^2 \left(-\frac{\pi}{6}\right) + 10(65) = 1406$.

16. By inspection, $x = 1$.

17. We have $\sin 20^\circ \left(\tan 10^\circ + \cot 10^\circ\right) = 2 \sin 10^\circ \cos 10^\circ \left(\frac{\sin 10^\circ}{\cos 10^\circ} + \frac{\cos 10^\circ}{\sin 10^\circ}\right) = 2(\sin^2 10^\circ + \cos^2 10^\circ) = 2$.

18. The polar graph is a circle with diameter 10. So the radius is 5 and the area is $25\pi$.

19. By inspection, $x = 1$ is a solution. Via Synthetic Division or favorite factoring method, $x^3 - 7x + 6 = (x - 1)(x - 2)(x + 3)$, so $x = 2$ is the other positive root. The desired product is therefore equal to 2.

20. We have $D \left(A + \sin \frac{C}{B}\right) = 2 \left(1 + \sin \frac{25\pi}{2}\right) = 2(1 + 1) = 4$.

21. Hexagonal numbers have a second-level common difference of $6 - 2 = 4$ and octagonal numbers have a second-level common difference of $8 - 2 = 6$. Thus, the hexagonal numbers are 1, 6, 15, 28, ... and the octagonal numbers are 1, 8, 21, 40, .... The answer is $15 + 40 = 55$.

22. Let $x = \cos t$ and $y = \sin t$; this is a legal substitution since $x^2 + y^2 = 1$. Note that $x + y = \cos t + \sin t = \sqrt{2} \sin \left(t + \frac{\pi}{4}\right)$, so the maximum value of $x + y$ is $\sqrt{2}$. Thus, the maximum value of $2(x + y)^3$ is $2 \left(2^{\frac{3}{2}}\right) = 2 \times 2\sqrt{2} = 4\sqrt{2}$.

23. All arguments are in degrees. We know that valid angles are of the form $53 + 360n$ or $127 + 360n$ for integer $n$. Setting $53 + 360n > 775$ yields $n > 2.00556$ ... so pick $n = 3$, resulting in an angle of 1133 degrees. However, setting $127 + 360n > 775$
yields $n > 1.8$, so pick $n = 2$, yielding an angle of $127 + 360(2) = 847$ degrees.

24. If $\log_k 3 \leq 4$, then $k^4 \geq 3$, or $k^8 \geq 9$, making the answer 9.

25. We have $(B + C)(A - \sum_{j=1}^{D+1} j) = (B + C)(55 - \sum_{j=1}^{9+1} j) = (B + C)(55 - 55) = 0$.

26. Notice that vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are mutually orthogonal to each other. Let $\mathbf{d} = [-6, -17, 6]$. We have $c_1 = \frac{\mathbf{d} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} = \frac{12}{2} = 6$, $c_2 = \frac{\mathbf{d} \cdot \mathbf{b}}{\mathbf{b} \cdot \mathbf{b}} = \frac{-68}{34} = -2$, and $c_3 = \frac{\mathbf{d} \cdot \mathbf{c}}{\mathbf{c} \cdot \mathbf{c}} = \frac{51}{17} = 3$, so $c_1c_2c_3 = -36$.

27. If $n$ is an integer, we have the identities $\sin(2\pi x) = \cos\left(\frac{\pi}{2} - 2\pi x + 2\pi n\right) = \cos\left(\frac{3\pi}{2} + 2\pi x + 2\pi n\right) = \cos(3\pi x)$, leading to the equations $3\pi x = \frac{\pi}{2} - 2\pi x + 2\pi n$ and $3\pi x = \frac{3\pi}{2} + 2\pi x + 2\pi n$. The first equation has solutions in the interval of .10, .50, 1.90, 1.30, and 1.70. The second equation has a single solution in the interval of 1.5. The sum of all these $x$-values is 6.

28. Forget what’s inside the sine function and focus on the outer coefficient. The amplitude is 2013.

29. The distance between the centers of the circles, $(3, 12)$ and $(-4, -12)$, is $\sqrt{7^2 + 24^2} = 25$. The sum of the radii of the circles is $12 + 13 = 25$. Thus, the circles intersect at exactly one point, which has area 0.

30. The determinant of $\begin{bmatrix} C & B \\ A & D \end{bmatrix}$ is $CD - AB = C(0) - (-36)(6) = 216$.

31. The coordinates of triangle $POQ$ are $(0, 0)$, $(5, 0)$, and $(x, y)$, where $x^2 + y^2 = 36$. The centroid of $POQ$ is the average of the coordinates, or $\left(\frac{x+5}{3}, \frac{y}{3}\right)$. Suppose $\left(\frac{x+5}{3}, \frac{y}{3}\right) = (a, b)$ so that $\frac{x+5}{3} = a$ and $\frac{y}{3} = b$. Solving each equation for $x$ and $y$, squaring both sides, and adding the equations, we arrive at $(3a - 5)^2 + (3b)^2 = 36$,
32. We want the angle opposite the side with length 6. Let this angle equal \( \theta \). By the Law of Cosines, 
\[
\cos \theta = \frac{7^2 + 8^2 - 6^2}{2 \times 7 \times 8} = \frac{11}{16}
\]
Thus, \( m + n = 11 + 16 = 27 \).

33. We have \( h(x) = f(x)g(x) = \sin x \cos x = \frac{1}{2} \sin(2x) \), so 
\[
h\left(\frac{\pi}{8}\right) = \frac{1}{2} \sin \frac{\pi}{4} = \frac{1}{2} \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}.
\]

34. Since \( 4096^2 = 4^{12} \), \( 8^4 = 2^{12} \), \( 81^3 = 3^{12} \), and \( 25^6 = 5^{12} \), the smallest element in \( D \) is \( 8^4 \), so \( f(m) = f(8^4) = 100 \).

35. We have 
\[
\frac{AD}{\pi} (\sin 15^\circ + C)^2 + B = \frac{(4\pi)(100)}{\pi} \left( \frac{\sqrt{6} - \sqrt{2}}{4} + \frac{\sqrt{2}}{4}\right)^2 + 27 = 400 \left( \frac{6}{16} \right) + 27 = 177.
\]

36. The equation is of the form \( M = PDP^{-1} \), where \( D \) is a diagonal matrix. Therefore, 
we have \( M^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1} \). Notice that 
\[
D^{10} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{10} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{10} = \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I
\]
is the identity matrix \( I \). Thus, \( M^{10} = PD^{10}P^{-1} = PIP^{-1} = PP^{-1} = I \). The sum of the elements of the \( 2 \times 2 \) identity matrix is \( 2 \).

37. Note that \( \frac{x}{100\pi} - 1 \) is less than \( -1 \) whenever \( x < 0 \) and greater than \( 1 \) when \( x > 200\pi \). Thus, the two graphs will have intersection points on the interval \( x \in [0, 200\pi] \), in which the graph of \( y = \sin x \) will exhibit 100 full cycles. We subtract 1 from this total to account for the double-counting of intersection points in the middle of the interval. The answer is \( 199 \).

38. Two consecutive angles of a parallelogram add up to \( \pi \). Thus, \( \cos \pi = -1 \).

39. The sum of the digits of \( 10^1 - 1 = 9 \times 1 = 9 \). The sum of the digits of \( (10 - 1)(10^2 - 1) = 891 \) is \( 9 \times 2 = 18 \). Basically, adding another term in the product in
accordance with the pattern creates a bunch of new digits whose sum is 9, with the number of “bundles” of digits whose sum is 9 equal to the number of 9s in the largest factor. Specifically, the sum of the digits of \((10 - 1)(10^2 - 1)(10^4 - 1)(10^8 - 1)\) is \(9 \times 8 = 72\) because \(10^8 - 1\) has eight 9s in its base-10 representation.

40. We have \(\frac{B-C}{A} + D = \frac{199-1}{2} + 72 = 172\).

41. Starting with quadrilateral \(ABCD\), draw auxiliary line segments to obtain the diagram above, where \(AF\) and \(DE\) are perpendicular to \(EG\), which contains the points \(C, B,\) and \(F\). Triangle \(ABF\) is a 45-45-90 triangle, so \(AF = BF = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}\).

Triangle \(DCE\) is a 30-60-90 triangle, so \(EC = 3\) and \(DE = 3\sqrt{3}\). Triangles \(AFG\) and \(DGE\) are similar. Therefore, \(\frac{AF}{DE} = \frac{FG}{EG}\), or \(\frac{\sqrt{3}}{3\sqrt{3}} = \frac{FG}{3+5-\sqrt{3}+\sqrt{3}+FG} = \frac{FG}{8+FG}\), or \(FG = 4\).

Moreover, \(AG = \sqrt{AF^2 + FG^2} = \sqrt{\sqrt{3}^2 + 4^2} = \sqrt{19}\). Based on the earlier equation, triangles \(AFG\) and \(DGE\) are in a 3-to-1 linear ratio, so \(AD = 2 \times AG = 2\sqrt{19}\).

42. Perhaps the fastest way to do this problem is to know in advance that \(\cos \frac{\pi}{5} = \frac{1+\sqrt{5}}{4}\); hence the minimal polynomial will also have \(\frac{1-\sqrt{5}}{4}\) as a root. The two roots have a
43. The exterior angles of a polygon add up to $2\pi$. Thus, $\sin(2\pi) = 0$.

44. The entry in the second row, second column of $M^{-1}$ is the second row, second column of the transposed adjoint matrix of $M$, divided by $|M|$. The second row, second column cofactor of $M$ is $(-1)^2 \begin{vmatrix} -1 & -5 \\ 4 & 5 \end{vmatrix} = 15$. Thus, $\frac{15}{4x-47} = 15$, so $x = 12$.

45. We have $A^2 + B^2 + C^2 - D^2 = (2\sqrt{19})^2 + 5^2 + 0^2 - 12^2 = -43$.

46. The numbers being plugged into the function are the first five positive perfect numbers. Recall that even perfect numbers have the form $g(x) = 2^{x-1}(2^x - 1)$, where $2^x - 1$ is prime; by inspection, the five smallest positive values of $x$ which makes this true are 2, 3, 5, 7, and 13. We have $f(g(x)) = \log_2(1 + \sqrt{8(2^{x-1}(2^x - 1))} + 1) - 2 = x - 1$. Thus, the answer is $(2 - 1) + (3 - 1) + (5 - 1) + (7 - 1) + (13 - 1) = 25$.

47. Suppose $\csc x = \cot x$. This leads to $\cos x = 1$ and $\sin x = 0$, hence $\csc x$ would be undefined and a triangle cannot be formed. Now suppose $\sec x = \csc x$. This leads to $\cos x = \sin x = 1/\sqrt{2}$, making a triangle with side lengths $\sqrt{2}$, $\sqrt{2}$, and 1. In particular, $\csc x = \sqrt{2}$. For the third case, $\sec x = \cot x$, we get the quadratic $\sin x = \cos^2 x = 1 - \sin^2 x$, which has the positive solution $\sin x = \frac{2}{1+\sqrt{5}}$ or $\csc x = \frac{1+\sqrt{5}}{2}$. This is the largest possible value of $\csc x$.

48. By the Power-Reducing Identities, $y = \sin^2(14x) = \frac{1-\cos(28x)}{2}$, which is a standard
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<tr>
<td>49.</td>
<td>There are three possible cases that are not necessarily disjoint:</td>
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<td>1. 53AB</td>
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<td>2. A53B</td>
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<td>3. AB53</td>
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<td>where A and B is any valid digit. Case 1 has $10 \times 10 = 100$ ways to occur, Case 2 has $9 \times 10 = 90$ ways to occur, and Case 3 has $9 \times 10 = 90$ ways to occur. Between these three cases, only Case 1 and Case 3 have a possibility of overlapping: the number 5353. By the Principle of Inclusion-Exclusion, the total number of possibilities is $100 + 90 + 90 - 1 = 279$.</td>
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<td>50.</td>
<td>We have $\frac{A(B^2 - B)}{c} + D$</td>
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