P1. Find the common ratio of the geometric sequence 6, −42, ...

P2. What is the area of the circle with equation \((x + 1)^2 + (y - 3)^2 - \frac{13}{\pi} = 0\)?

P3. If the probability of event E happening is \(\frac{5}{6}\), what are the odds against E happening? Express your answer as a common fraction.

P4. If \(\theta\) is an acute angle such that \(5 \sin \theta = 3\), find the value of \(\sec \theta\) as a common fraction.

P5. Let \(A, B, C,\) and \(D\) be the answers to questions P1, P2, P3, and P4, respectively. Evaluate: \(\frac{AB}{CD}\)
1. Find $x$ as a common fraction: $4 + \sqrt{10 - x} = 6 + \sqrt{4 - x}$

2. Find the amplitude of the graph $y = 2 \cos x - 2\sqrt{3} \sin x$.

3. If $\csc x = \frac{13}{\sqrt{7}}$ find $169 \cos(2x)$.

4. Solve for $x$: $\log_2(2x) + \log_4 x + \log_8 x = 12$

5. Let $A, B, C,$ and $D$ be the answers to problems 1, 2, 3, and 4, respectively. Evaluate: $AB + C + \sqrt{D}$
6. For integer $n$, let $\tau(n)$ equal the number of positive divisors of $n$. How many integers $N \in (0,200)$ satisfy the congruence $\tau(N) \equiv 1 \pmod{2}$?

7. If $x$ is a real number, find the number of solutions to $x + \sin x + e^x = 2$.

8. Evaluate: $2(\cos^2 0^\circ + \cos^2 1^\circ + \cos^2 2^\circ + \cdots + \cos^2 89^\circ + \cos^2 90^\circ)$

9. What is the remainder when $2x^{603} - 3x^{250} + 10 - 6x^{25}$ is divided by $x + 1$?

10. Let $A$, $B$, $C$, and $D$ be the answers to problems 6, 7, 8, and 9, respectively. Evaluate: $\sqrt{A} + 2 + \sqrt{C} + D - 2B$
11. Find, as a common fraction, the sum of all real numbers $x$ such that $2x^3 + x^2 - 4 = 8x$.

12. Find the sum of the solutions to $\sin^2(5\theta) + \sin(2\theta) + \cos^2(5\theta) = 1$, where $\theta \in (\pi, 5\pi]$.

13. In triangle $ABC$, $\angle C = \frac{\pi}{2}$ and $\angle B = \theta$. If $\sec \theta = \frac{5}{3}$ and $|AB| = 15$, find the area of $ABC$.

14. What is the total surface area of a regular octahedron of volume $4/3$?

15. Let $A, B, C,$ and $D$ be the answers to problems 11, 12, 13, and 14, respectively. Evaluate: $A \tan^2 B + CD^2$
16. How many integers \( x \) satisfy \( |x| - 7| \leq 8? \)

17. The line with equation \( 2x - ky = 2013 \) makes a \( 30^\circ \) angle with the positive \( x \)-axis. Find \( k^4 \).

18. Let \( A, B, \) and \( C \) be the angle measures of a triangle. Let \( M \) be the maximum value of \( \sin A \sin B \sin C \). Find the value of \( 128M^2 \).

19. Find the product of all distinct complex numbers \( z \) with positive real part and \( z^6 = -64 \).

20. Let \( A, B, C, \) and \( D \) be the answers to problems 16, 17, 18, and 19, respectively.
Evaluate: \( A + \sqrt{B} + \frac{2C}{D} \)
21. Find the area of a quadrilateral with side lengths of 39, 52, 25, and 60 in that order.

22. A cube has volume of \( \cos^3 x \) (where \( 0 < x < \frac{\pi}{2} \)) and surface area of \( \frac{36}{17} \). If \( \sin^2 x = \frac{m}{n} \), where \( m \) and \( n \) are positive relatively prime integers, find \( m + n \).

23. Find the number of degrees of the angle coterminal to \( 6912^\circ \) in the interval \( (0^\circ, 360^\circ) \).

24. If \( M \) and \( N \) are positive perfect cubes less than 1000 such that \( M - N = 169 \), find \( M^{\frac{1}{3}} + N^{\frac{1}{3}} \).

25. Let \( A, B, C, \) and \( D \) be the answers to problems 21, 22, 23, and 24, respectively. Evaluate: \( A - B + C - D^2 \)
26. Let $M$ be a $4 \times 4$ matrix such that $M \times \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ c/2 \\ 3d \\ a/4 \end{bmatrix}$ for all real numbers $a, b, c,$ and $d$. Find the sum of the elements of $3M^{-1}$.

27. The domain of $f(x) = \sin^6 x + \cos^6 x$ is all real numbers $x$. The range of $f$ is the interval $I = [a, b]$. Find the midpoint of $I$.

28. Find the number of petals in the polar graph $r = \sin(24\theta)$.

29. Find the distance from $(0,0)$ to the focus of the parabola with equation $8x + y^2 = 6y - 25$.

30. Let $A, B, C,$ and $D$ be the answers to problems 26, 27, 28, and 29, respectively.
Find the units digit of $\left(\frac{CB}{D}\right)^A$. 

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Round #6 Alpha State Bowl  
Mu Alpha Theta National Convention 2013
31. Find $P(100)$, where $P(x)$ is a polynomial with real coefficients and $P(x^2) + 2x^2 + 10x = 2xP(x + 1) + 3$ for all real $x$.

32. A triangle inscribed in the unit circle has angles measuring $\alpha, \beta,$ and $\gamma$. The perimeter of the triangle is 5. Evaluate: $\sin \alpha + \sin \beta + \sin \gamma$

33. Calculate $\arcsin(\sin 40^\circ + \sin 20^\circ)$ and express your answer in degrees. Recall that $-90^\circ \leq \arcsin u \leq 90^\circ$ for $u \in [-1, 1]$.

34. The sequence $17, 20, 25, 32, \ldots$ has $n$th term given by $a_n = n^2 + 16$. Find the largest possible value of the greatest common divisor of two consecutive terms of this sequence as $n$ ranges across the positive integers.

35. Let $A, B, C,$ and $D$ be the answers to problems 31, 32, 33, and 34, respectively. Evaluate: $A + BC + D$
36. In triangle \(ABC\) with centroid \(P\), let \(D\) and \(E\) be the foot of the medians to sides \(BC\) and \(AC\), respectively. If \(AP\) is perpendicular to \(BE\), \(|AD| = 6\), and \(|BE| = 9\), find the area of \(ABC\).

37. Find the number of times the polar graph \(r = 2\theta \pi\) intersects the line segment whose endpoints are the Cartesian coordinates \((\sqrt{2}, \sqrt{2})\) and \((64\sqrt{2}, 64\sqrt{2})\).

38. Two sides of a triangle have length 8 and 15, while the sine of the acute angle between them is \(\frac{8}{17}\). The measure of this angle is doubled while keeping the two side lengths the same, resulting in a new triangle. What is the ratio of the area of the \textit{old triangle} to the new triangle? Express your answer as a common fraction.

39. Let \(a\) be a sequence such that \(a_1 = 2\) and \(a_n(1 - a_{n+1}) = 1\) for \(n \geq 1\). Evaluate: \(\sum_{n=1}^{833} a_n\)

40. Let \(A\), \(B\), \(C\), and \(D\) be the answers to problems 36, 37, 38, and 39, respectively. Evaluate: \(A + \frac{D}{30BC+2}\)
41. Define $\Pi(S)$ as the product of the elements of a set $S$. Let $S_1, S_2, S_3, \ldots, S_{15}$ be the nonempty subsets of $S = \{1, 2, 3, 4\}$. Evaluate: $\sum_{n=1}^{15} (\Pi(S_n))^{-1}$

42. Find, in degrees, the measure of the smallest angle in a right triangle with legs of length $a$ and $b$ and hypotenuse of length $2\sqrt{ab}$, where $a$ and $b$ are positive numbers.

43. If $\sin u = \frac{3}{4}$, $\cos v = -\frac{1}{7}$ and $\tan w = 28$, evaluate: $12\sin(-u) - .5 \cos(-v) \tan(-w)$

44. Let $P$ be a point inside square $ABCD$ such that $|AP| = 5$, $|BP| = 2\sqrt{2}$, and $|CP| = 3$. Find the area of $ABCD$.

45. Let $A$, $B$, $C$, and $D$ be the answers to problems 41, 42, 43, and 44, respectively. Evaluate: $10D - A^B + C$
46. Find the sum of all positive integers \( n \) such that \( \frac{2210}{(3n+5)(2n+3)} \) is an integer.

47. Find the smallest positive angle \( x \) (in radians) satisfying the equation
\[
\sin\left(\frac{2x}{3}\right) \cos\left(\frac{4x}{3}\right) + \cos\left(\frac{2x}{3}\right) \sin\left(\frac{4x}{3}\right) \left( \cos\left(\frac{16x}{5}\right) \cos\left(\frac{6x}{5}\right) + \sin\left(\frac{16x}{5}\right) \sin\left(\frac{6x}{5}\right) \right) = \frac{1}{4}.
\]

48. If \( M = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \), find the sum of the squares of the elements of \( M^{2013} \).

49. Let \( f(x) \) denote the integer closest to \( \sqrt{x} \). Evaluate: \( \sum_{n=1}^{650} \frac{1}{f(n)} \)

50. Let \( A, B, C, \) and \( D \) be the answers to problems 46, 47, 48, and 49, respectively. Evaluate: \( AC + \frac{\pi}{B} + D \)