# Alpha Trigonometry

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<th>Problem</th>
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| 1       | Evaluate: $\cot \frac{\pi}{6}$ | C - $\sqrt{3}$ | By definition, $\cot x = \frac{\cos x}{\sin x}$. Evaluating this for $x = \frac{\pi}{6}$, we obtain that $\cos x = \frac{\sqrt{3}}{2}$ and $\sin x = \frac{1}{2}$. Thus, $\cot x = \sqrt{3}$.
| 2       | How many petals does the polar graph of $r = 43 \cos(2013\theta)$ have? | C - 2013 | The number of petals on this graph is determined by the coefficient of theta. Since the coefficient is odd, the number of petals is just equal to the coefficient, 2013.
| 3       | Which of the following angles in radians is equivalent to 36 arcseconds? | B - $\frac{\pi}{18000}$ | There are 60 arcminutes in a degree and 60 arcseconds in an arcminute. Recall that there are $\frac{\pi}{180}$ radians to a degree, thus 36 arcseconds is equivalent to $\frac{\pi}{18000}$.
| 4       | Express the following in terms of cosine functions: $\sin(\arctan(\cos x))$. | D - $\frac{\cos x}{\sqrt{1 + \cos^2 x}}$ | A useful diagram is a right triangle with angle $x$, whose opposite leg is of length $\cos x$ and adjacent leg of length 1 and the hypotenuse of length $\sqrt{1 + \cos^2 x}$. From this triangle we find that $\sin x = \frac{\cos x}{\sqrt{1 + \cos^2 x}}$.
| 5       | Which of the following trigonometric functions is odd? | B - $\sin 2x$ | A function $f(x)$ is odd if the following property holds: $f(-x) = -f(x)$.
| 6       | Two rays form an angle in the Cartesian plane. If one of the ray | A - $-\frac{5}{13}$ | Use the trigonometric identity: $\tan^2 x + 1 = \sec^2 x$ to solve for the cosine. We take the negative solution as the angle is in... |
is the positive x-axis, the other ray points towards Quadrant III, and the tangent of the resulting $\frac{12}{5}$ angle is $\frac{12}{5}$, what is the cosine of this angle?

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<th>7</th>
<th>What is the sum of the solutions of the equation $\tan \theta = \sqrt{3}$, in the domain $(-2\pi, 2\pi)$?</th>
<th>D - $\frac{2\pi}{3}$</th>
<th>The solutions to the equation are of the form $\theta = \frac{\pi}{3} + k\pi$ for integer values of $k$. Summing the solutions in the specified domain yields D.</th>
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<td>8</td>
<td>The sides of a triangle measure 65, 72, and 97. This triangle is:</td>
<td>C - Right</td>
<td>The sides of this triangle satisfy the Pythagorean identity. Hence the sides form a right triangle.</td>
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<td>9</td>
<td>Restricted to $[0, 2\pi)$, what is the argument of $3+i\sqrt{3}$? Note: $i = \sqrt{-1}$.</td>
<td>C - $\frac{\pi}{6}$</td>
<td>Rewriting the complex number in polar form, we find that the argument (or polar angle) is C.</td>
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<td>10</td>
<td>What is the cosine of the angle between the complex vectors $1+2i$ and $2-3i$?</td>
<td>A - $\frac{4\sqrt{65}}{65}$</td>
<td>The cosine of the angle between any two complex vectors is defined as: $\cos \theta = \frac{\text{Re}(a \cdot b)}{</td>
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<td>11</td>
<td>A helix is parameterized by $x = 4\cos t$, $y = 3\sin t$, and $z = t$. What is the distance between the points at time $t = 0$ and $t = \pi$?</td>
<td>C - $\sqrt{64 + \pi^2}$</td>
<td>The points in question are $(4,0,0)$ at time $t = 0$ and $(-4,0,\pi)$ at time $t = \pi$.</td>
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<td>12</td>
<td>Express the line $3x + 5y = 7$ in polar coordinates.</td>
<td>A - $r = \frac{7}{3 \cos \theta + 5 \sin \theta}$</td>
<td>Use the change of variables: $x = r \cos \theta$ and $y = r \sin \theta$.</td>
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<td>Simplify $\sum_{n=0}^{\infty} 2 \cos^n \theta$ for $0 \leq \theta &lt; \pi / 2$.</td>
<td>$D - \csc^2 \theta$ Realize that the series is an infinite geometric one. Thus the sum, given the proper domain for convergence, converges to $\frac{2}{1 - \cos \theta}$, which after using the double angle formula for cosine reduces to $D$.</td>
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<tr>
<td>What is the maximum value of the function: $y = \cos^4 x - \sin^2 x$?</td>
<td>$B - f(x) \leq 1$ It may help to draw the two functions $y = \cos^4 x$ and $y = \sin^2 x$. It is clear that the maximum difference between these two occurs when one reaches a maximum value of one and the other a minimum value of 0.</td>
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<td>What is the period of the graph of $y = \sin^5 x + \sin^2 x$?</td>
<td>$C - 2\pi$ First function has period $\pi$. Second function has period $2\pi$. Sum yields a period of $2\pi$.</td>
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<td>One of the first trigonometric functions is the chord function, defined as: $\text{crd}(\theta) = 2 \sin \frac{\theta}{2}$. What is $\cos^2 \theta$ in terms of this ancient function?</td>
<td>$D - 1 - 4\text{crd}(\theta) + 4\text{crd}^2(\theta)$ Using the half-angle formula for sine, we can solve for cosine in terms of the chord function.</td>
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<td>What is the entry in the first row, first column of the matrix $M^2$, for $M = \begin{bmatrix} \cos \theta &amp; -\sin \theta \ \sin \theta &amp; \cos \theta \end{bmatrix}$?</td>
<td>$A - 1 - 2\sin^2 \theta$ By matrix multiplication, the entry is $\cos^2 x - \sin^2 x = 1 - 2\sin^2 x$.</td>
<td></td>
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<td>The trigonometric ratio $\sec \frac{\pi}{12}$ can be expressed in the form of</td>
<td>$C - \sqrt{2}$ We begin with evaluating cosine at the angle $\frac{\pi}{12}$ using the half angle formula for cosines.</td>
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\[a\sqrt{b} + c\sqrt{d}\] where \(a, b, c, d\) are integers and \(a\) even. What is the product \(abcd\)?

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<tr>
<th>[\cos \frac{\pi}{12} = \cos \frac{\pi}{6}]</th>
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<td>[= \sqrt{1 + \cos \frac{\pi}{6}}]</td>
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<tr>
<td>[= \frac{1}{2} \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}]</td>
</tr>
<tr>
<td>[= \frac{1}{2} \sqrt{2 + \sqrt{3}}]</td>
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Thus, the secant of \(\frac{\pi}{12}\) is

\[\sec \frac{\pi}{12} = \frac{1}{\cos \frac{\pi}{12}}\]

\[= \frac{2}{\sqrt{2 + \sqrt{3}}}\]

\[= 2\sqrt{2 - \sqrt{3}}\]

Hence,

\(a = 2\) \(c = -1\)

\(b = 4\) \(d = 3\)

So the product is \(abcd = -24\)

19. What is the value of \(\sum_{n=2}^{6} e^{\frac{n}{2}i}\)?

Note: \(i = \sqrt{-1}\).

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<tr>
<th>(= e^{\frac{2}{2}i} + e^{\frac{3}{2}i} + e^{\frac{4}{2}i} + e^{\frac{5}{2}i} + e^{\frac{6}{2}i})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(= -1 + -1 + 1 + 1 - 1)</td>
</tr>
<tr>
<td>(= -1)</td>
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20. What is the exact value of \(\tan 75^\circ\)?

| \(= 2 + \sqrt{3}\) |

Notice that the argument of the tangent function is a sum of 45° and 30°. We can either recall the angle sum formula for tangent or use the definition of tangent as a ratio of the sine and cosine functions and their respective angle sum.
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<td>21</td>
<td>Triangle XYZ has side lengths of $x = 5$, $y = 6$, and $z = 4$. What is the exact value of $\tan \frac{Y}{2}$?</td>
<td>$A - \frac{\sqrt{7}}{3}$</td>
<td>Using the law of cosines, we have that $\cos Y = \frac{1}{8}$. This will help when we consider the half angle formula for sine and cosine.</td>
</tr>
<tr>
<td>22</td>
<td>What is the range of $f(x) = \cot^{-1} x$?</td>
<td>$E - \text{NOTA}$</td>
<td>There are multiple possibilities for the range of arccotangent function and all are equally appropriate.</td>
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<tr>
<td>23</td>
<td>What is the probability that a randomly chosen angle $\theta \in [0, 2\pi)$ satisfies the inequality: $\cos 2\theta &lt; \sin \theta$?</td>
<td>$C - \frac{1}{3}$</td>
<td>Using the double angle formula for cosine, we have that the inequality can be rewritten as $2\sin^2 \theta + \sin \theta - 1 &lt; 0$. This inequality holds for when $\sin \theta &gt; \frac{1}{2}$, or $\theta \in \left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$.</td>
</tr>
<tr>
<td>24</td>
<td>Solve for smallest positive value of $x$ such that $\cos 3x + 3\cos x - 4 = 0$.</td>
<td>$E$</td>
<td>Multiples of $2\pi$ form the solution set for this equation.</td>
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<tr>
<td>25</td>
<td>How many distinct obtuse triangles can be made by selecting three distinct integers between 1 and 10, inclusive, as side lengths?</td>
<td>$E$</td>
<td>There are 15 acute triangles, 33 obtuse triangles, and 2 right triangles.</td>
</tr>
<tr>
<td>26</td>
<td>Given the triangle ABC, $a = 4$,</td>
<td>$C - 12 - 4\sqrt{3}$</td>
<td>Drawing an altitude, we have that the area of the triangle</td>
</tr>
</tbody>
</table>
1. \( m \angle A = 75^\circ, m \angle B = 60^\circ \), find the area of the triangle.

2. ABC is \( 2b \sin C \). The angle C can be determined using the property that the sum of the interior angle of any triangle is 180 degrees. The side length \( b \) can be determined by using the law of sines and the angle sum formula for sines.

27. Which of the following expressions has the smallest value?

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<th>A - ( \cos 1 )</th>
<th>Drawing a picture may help distinguish whether the sine function or cosine function is larger at 1 radian.</th>
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28. Which of the following expressions are equivalent to \( \cos x + \sin x \) for all values of \( x \)?

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<tr>
<th></th>
<th>A - ( \sqrt{2} \cos \left( \frac{\pi}{4} + x \right) )</th>
<th>We begin by using the following definition for the cosine function: ( \cos x = \sin \left( \frac{\pi}{2} - x \right) ). Hence,</th>
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<td>( \cos x + \sin x = \sin \left( \frac{\pi}{2} - x \right) + \sin x )</td>
<td>( = 2 \sin \left( \frac{\pi}{4} \right) \sin \left( \frac{\pi}{4} - x \right) )</td>
</tr>
<tr>
<td></td>
<td>( = \sqrt{2} \cos \left( \frac{\pi}{4} + x \right) )</td>
<td>where the second equality comes from the summation identity: ( \sin x + \sin y = 2 \sin \left( \frac{x+y}{2} \right) \sin \left( \frac{x-y}{2} \right) ), and the third from evaluating the sine function and applying the definition used earlier.</td>
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29. Solve for \( x \): \( \cos^{-1} x = \sin^{-1} 3x \).

|   | A - \( \frac{\sqrt{10}}{10} \) | Take the cosine of both sides. The left side will reduce to \( x \), while the right side can be calculated by considering the cosine of an angle whose sine ratio is 3x. |

30. Which of the following polar coordinates are equivalent to the rectangular coordinates

| B - \( 4, \frac{\pi}{5} \) | From the solutions, we are inspired to consider multiples of \( x = \frac{\pi}{5} \). Drawing a picture can help reveal that \( \cos 2x = -\cos 3x \). Using the angle sum formula for cosines, we |
\[
\left(\sqrt{5} + 1, \sqrt{10 - 2\sqrt{5}}\right)
\]

find that \( x = \frac{\pi}{5} \) will indeed yield rectangular coordinates that are a quarter of the length of the requested coordinate pair.