2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

1. For all solutions, assume the longer dimension is horizontal and the shorter dimension is vertical.

a. 2x2: There are no ways to get three squares in a row, so the answer is 0.

b. 2x3: There are three cases: sideways (four ways), vertical on an edge (two ways), and vertical in the middle (one way). For these cases, the probabilities are $\frac{1}{3}$, 0, and 0, for an answer of $\frac{4}{7} \times \frac{1}{3} = \frac{4}{21}$.

c. 3x3: Because of symmetry, we only need to consider sideways pairs. The cases are in a corner (four ways), and in the center (two ways), with probabilities of $\frac{1}{3}$ and $\frac{1}{5}$, respectively, for an answer of $\frac{4}{6} \times \frac{1}{3} + \frac{2}{6} \times \frac{1}{5} = \frac{2}{9} + \frac{1}{15} = \frac{13}{45}$.

d. 3x4: The cases are sideways in a corner (four ways), sideways in the center of a 4 side (two ways), sideways in the center of a 3 side (two ways), sideways in the very center (one way), vertical in a corner (four ways), and vertical not in a corner (four ways), with probabilities of $\frac{1}{3}, \frac{1}{2}, \frac{1}{5}, \frac{1}{3}, \frac{1}{3}$, and $\frac{1}{5}$, for an answer of $\frac{4}{17} \times \frac{1}{3} + \frac{2}{17} \times \frac{1}{2} + \frac{2}{17} \times \frac{1}{5} + \frac{1}{17} \times \frac{1}{3} + \frac{4}{17} \times \frac{1}{3} + \frac{4}{17} \times \frac{1}{5} = \frac{9}{51} + \frac{1}{17} + \frac{6}{85} = \frac{78}{255} = \frac{26}{85}$.

e. 4x4: Because of symmetry, we only need to consider sideways pairs. The cases and probabilities are in a corner (four ways, $\frac{1}{3}$), along the center of a side (two ways, $\frac{1}{2}$), perpendicular to the center of a side (four ways, $\frac{1}{5}$), and next to the center (two ways, $\frac{1}{3}$), for an answer of $\frac{4}{12} \times \frac{1}{3} + \frac{2}{12} \times \frac{1}{2} + \frac{4}{12} \times \frac{1}{5} + \frac{2}{12} \times \frac{1}{3} = \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \frac{1}{18} = \frac{57}{180} = \frac{19}{60}$.

f. 4x5: The cases and probabilities are sideways in a corner (four ways, $\frac{1}{3}$), sideways along a side (four ways, $\frac{1}{2}$), sideways perpendicular to a side (four ways, $\frac{1}{5}$), sideways in the interior (four ways, $\frac{1}{3}$), vertical in a corner (four ways, $\frac{1}{3}$), vertical along a side (two ways, $\frac{1}{2}$), vertical perpendicular to a side (six ways, $\frac{1}{5}$), and vertical in the interior (three ways, $\frac{1}{3}$), for an answer of $\frac{4}{31} \times \frac{1}{3} + \frac{4}{31} \times \frac{1}{2} + \frac{4}{31} \times \frac{1}{5} + \frac{4}{31} \times \frac{1}{3} + \frac{4}{31} \times \frac{1}{3} + \frac{2}{31} \times \frac{1}{2} + \frac{6}{31} \times \frac{1}{5} + \frac{3}{31} \times \frac{1}{3} = \frac{15}{93} + \frac{6}{62} + \frac{10}{155} = \frac{5}{31} + \frac{3}{31} + \frac{2}{31} = \frac{10}{31}$.

g. 5x5: Because of symmetry, we only need to consider sideways pairs. The cases and probabilities are in a corner (four ways, $\frac{1}{3}$), along a side (four ways, $\frac{1}{2}$), perpendicular to a side (six ways, $\frac{1}{5}$), and interior (six ways, $\frac{1}{3}$), for an answer of $\frac{4}{20} \times \frac{1}{3} + \frac{4}{20} \times \frac{1}{2} + \frac{6}{20} \times \frac{1}{5} + \frac{6}{20} \times \frac{1}{3} = \frac{1}{15} + \frac{1}{10} + \frac{3}{50} + \frac{1}{10} = \frac{49}{150}$.

h. 5x∞: The cases and probabilities are sideways on an edge (two ways, $\frac{1}{2}$), sideways in the interior (three ways, $\frac{1}{3}$), vertical on an edge (two ways, $\frac{1}{5}$), and vertical in the interior (two ways, $\frac{1}{3}$), for an answer of $\frac{2}{9} \times \frac{1}{2} + \frac{3}{9} \times \frac{1}{3} + \frac{2}{9} \times \frac{1}{5} + \frac{2}{9} \times \frac{1}{3} = \frac{1}{9} + \frac{1}{9} + \frac{2}{45} + \frac{2}{27} = \frac{46}{135}$.

i. $\infty \times \infty$: All pairs are identical, giving an answer of $\frac{1}{3}$.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

In general, if there are at least 3 rows and columns, we can perform the following analysis:

Let's start with the first and last row, which are the same. There are 2 ends both with probability 1/3, and the (n-3) pairs in the middle each have probability 1/2. Since the first and last are the same, multiply this sum by 2.

$$2\left[2\left(\frac{1}{3}\right)+(n-3)\left(\frac{1}{2}\right)\right]$$

The m-2 middle rows each have an end with probability 1/5 and (n-3) middle pairs with probability 1/3.

$$(m-2)\left[2\left(\frac{1}{5}\right)+(n-3)\left(\frac{1}{3}\right)\right]$$

This sum simplified to:

$$\frac{5mn+5n-9m-7}{15}$$

Now the columns follow the same analysis with m and n reversed. In total, there are m(n-1) rows pairs and n(m-1) column pairs. The probability we want is:

$$\frac{(5mn+5n-9m-7)+(5mn+5m-9n-7)}{15[m(n-1)+n(m-1)]}=\frac{10mn-4n-4m-14}{15[2mn-m-n]}$$

This gives the answers to parts c-g quickly, and you can take limits as *m* and/or *n* go to infinity to get parts h and i.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

2. In how many distinguishable ways can the letters in the "words" below be arranged so that no letter appears to be in the same location as it originally was?

a. ABCDEFG

a. $7! - 7\times265 - 21\times44 - 35\times9 - 35\times2 - 21\times1 - 0 - 1 = 1854$ Everyone should be able to derive this or look it up.

b. ABCDEFA

b. The A's can be arranged $5c2 = 10$ ways. Let's assume they went to the E- and F-positions, in which case E and F can go anywhere, for $5\times4 = 20$ ways. 2 of those ways use the two A-positions, 12 of those ways use one A-position, and 6 of those ways use no A-positions. Those three cases respectively have 2, 3, and 4 ways to arrange the remaining letters, for an answer of $10(2\times2 + 12\times3 + 6\times4) = 10\times64 = 640$.

c. ABCDEAA

c. The three A's needs to go on three of BCDE (4 ways). The fourth one needs to not go on itself (3 ways), after which the other three can go on any remaining space ($3! = 6$ ways), for an answer of $4\times3\times6 = 72$.

d. ABCDEAB

AA could go to BB (1 way), BX (6 ways), or XX (3 ways). BB could do similarly as regards the A's. Combining the two, if the AA's & BB's switch places there are $1\times1\times2 = 2$ possible arrangements of the other three. If AA goes to BB while BB goes to AX, there are $1\times6\times3 = 18$ arrangements (as there are if the A's & B's switch behaviors). If AA goes to BB while BB goes to XX, there are $1\times3\times4 = 12$ arrangements (same if you switch the A's & B's). If AA goes to BX and BB goes to AX, there are $6\times4\times4 = 96$ arrangements (note that AX had fewer options when combined with BX). If AA goes to BX while BB goes to XX, there are $6\times1\times6 = 36$ arrangements (note that XX had fewer arrangements when combined with AX, and that there is a switched case for this arrangement). There is no way for AA to go to XX and BB to go to XX as well, so the total number of arrangements is $2 + 18 + 18 + 12 + 12 + 96 + 36 + 36 = 230$.

e. ABCDABC

AA could go to BB, BC, BD, CC, or DC. Let's call those cases Z, Y, X, W, and V. Similarly, BB could go to AA, AC, AD, CC, or CD (U-Q), and CC could go to AA, AB, AD, BB, or BD (P-L). We can combine these in any way that doesn't use more than 2 of A, B, or C and uses D exactly once (so that D will go elsewhere). These combinations are ZTN, ZRN, ZQP, YUL, YTN, YTL, YSO, YQP, YQO, XTO, XRP, XRO, WUL, WSO, WSM, VUM, VTO, and VTM, with 4, 2, 2, 4, 8, 8, 8, 4, 8, 8, 2, 4, 2, 4, 2, 2, 8, and 4 respective arrangements, for a total of 84.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

f. ABCDAAB

f. There are $4c3 = 4$ ways to arrange the A's, 2 of which leave a B-position unfilled and 2 of which leave a C- or D-position unfilled. In the first case, there are 2 non-B possibilities for the B-position and $3c2 = 3$ ways to fill the A positions. In the second case, there are 2 ways to fill the C- or D-position, one of which leaves two B's to place and one of which leaves one B to place. For the first sub-case, there are $3c2 = 3$ ways to fill the A-positions, while for the second case there are $3! = 6$ ways to fill the A-positions. This gives an answer of $2\times2\times3 + 2\times1\times3 + 2\times1\times6 = 12 + 6 + 12 = 30$.

g. ABCAABB

g. There are 6 places the C could go, after which the A's and B's are uniquely determined.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

3. In Hasbro's board game, Battleship:

a. If you play to minimize the maximum number of turns it could take you to eliminate your opponent's fleet, what is the largest number of turns this can take (assume you are not defeated)?

Our worst-case scenario is that we find the patrol boat last. Because of this, we should not leave any 2-unit space while we're guessing in advance of a hit, or else later we'll have to go back to each 2-unit space and guess in it later. So, our initial pattern should be to hit every other space in a checkerboard pattern (A2, A4, …), which is 50 guesses. We'll eventually hit every ship this way, but must be prepared for that to happen only in our last many shots. Actually, if we still haven't hit anything by I4 (42 shots so far), we'll know that we can sink every ship in another 17 shots, for a total of 59 shots. However, I suspect that it's actually worse for us if we find the ships one at a time, as then we have to shoot around trying to find the rest of each ship.

I suspect that hitting a ship with a shot that is adjacent to an edge is an advantage, so let's assume we find the first ship when we hit it at C4.

Should we shoot right next to C4, or should we shoot two away to keep our search pattern?

If we try right next to, we'll shoot at C5 or D4. The worst thing that could happen would be a miss, in which case we'll shoot at the other one, get a hit (because a miss would be great news, so we have to plan for that not to happen), continue in that direction until we miss, then go back to get the last part of the ship. No matter which ship it is, we got hits that we needed to eventually get, and we also got two misses we didn't necessarily have to get. If the ship was a 3 or 5, there is one miss that wouldn't have been in our search pattern; if it was the 4, there are two such misses. If it was a 3, only our additional hit was a location that would have been in our original search pattern. If it was a 4 or 5, a second hit would have been in our search pattern. Once we've done this, we need to go back to our original guessing grid, even if that means we shoot right next to one of our current shots; otherwise there will be some regions that could hide a 2. The worst case is that we find the biggest ones first and the smallest ones last. We'll eventually hit all 50 in our search pattern, plus $3 + 2 + 2 + 2 + 1 = 10$ hits that weren't part of the search pattern, plus $1 + 2 + 1 + 1 + 2 = 7$ misses that wouldn't have been in our search pattern, for a current answer of 67, which is indeed more than the 59 above.

If instead we'd tried two-away, we'd have shot at C6 or E4. I'm not sure a miss is so bad, as the miss is just part of our original search pattern. Let's assume a miss is bad, however. If it's a 5, we shoot at the other spot and get a hit, so we shoot in between and get a hit, so we shoot F4 and get a hit, then we shoot G4 and get a miss, and finally we shoot B4 and finish off the boat. If it's a 4, the same process occurs, except that we get our miss at F4. If it's a 3 (two ships), we get a miss, a miss, a closer miss, and then two hits. If it's the 2 at the very end, we get a hit at J9, then a miss, a miss, and a hit. In the end, we shot all 50 of our initial search grid, $3+2+2+2+1=10$ hits that weren't on the grid, and $0+1+1+1+2=5$ misses that weren't on the grid, for an answer of 65, which is better than 69, so this is the better strategy and is our answer.

b. "Everyone knows" that you shouldn't have your ships next to each other, not even diagonally. What is the smallest area of a rectangular region of the board that can contain all of your ships subject to this constraint?

You can fit all the ships in two rows with a blank row in between, for an answer of 30.

c. How many arrangements of the ships are possible within such a smallest-perimeter rectangle? Do not count reflections or rotations of an arrangement as different.

The two columns must be a 5 & 4 and a 2 & two 3's. The 5 & 4 only have one arrangement, as the other arrangement will just be a reflection. The two columns only have one arrangement, as the other arrangement will just be a reflection. The 233 column has three arrangements relative to the other column. Thus, the answer is $1\times1\times3=3$.

d. Using the same constraints as in b, what is the smallest perimeter of such a rectangle?

You can fit all the ships into a 7x5 rectangle, for an answer of 24.

e. How many arrangements of the ships are possible within such a smallest-perimeter rectangle? Do not count reflections or rotations of an arrangement as different.

Only a 7x5 rectangle works; 6x6, 8x4, 9x3, 10x2, and 11x1 don't work. Assuming the 5-direction to be vertical, the 5 can be vertical on an edge, vertical in the third column, horizontal in a corner, horizontal along the middle of an edge, horizontal in the third row on an edge, or horizontal in the center (six cases).

Working backwards, if the 5 is in the center, the 4 & 2 must share a row (one way) and the 3 & 3 must share a row (one way), for a subtotal of 1 arrangement.

If the 5 is in the third row on an edge, the 4 can be in an adjacent corner (11 ways) or one away from an adjacent corner (8 ways), for a subtotal of 12 arrangements.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

If the 5 is horizontal along the middle of an edge, the 4 & 2 must share a row (2 ways to pick which row, one way to arrange them) and the 3 & 3 must share a row (one way), for a subtotal of 2 arrangements.

If the 5 is horizontal in a corner, the 4 can be parallel in the middle of the near edge, one over from there, parallel in the middle of the far edge, parallel in the near corner, two over from there, parallel in the far corner, or vertical on the far edge. Working backwards within this group, if the 4 is vertical on the far edge (two ways), the 3, 3, & 2 must fit in a 3x5 rectangle, which can happen in 18 ways relative to other ships. This will come up a few times later in this analysis. In general, to analyze cases with many arrangements of the 3's and 2, we considered two horizontal 3's, two vertical 3's, and one of each. This gives a subtotal of $2\times 18 = 36$ arrangements for this case. If the 4 is in the far corner, there are 11 ways. If the 4 is one over from there, there are 6 ways. If the 4 is in the near corner, there are 4 ways. If the 4 is in the middle of the far side, there is only 1 way. If the 4 is two away from there, there are 8 ways. If the 4 is in the middle of the adjacent side, there are 9 ways. This gives a subtotal of $36 + 11 + 6 + 4 + 1 + 8 + 9 = 75$ arrangements.

If the 5 is vertical in the third column, the 4 must be against the near edge, leaving a 3x5 rectangle for 18 arrangements.

If the 5 is vertical on an edge, the 4 can be parallel in the third column, parallel on the far edge, or horizontal on an edge (two ways). If the 4 is parallel in the third column, there are 18 ways. If the 4 is parallel on the far edge, there are 18 ways. If the 4 is horizontal on an edge, there are 18 ways. This gives a subtotal of $18 + 18 + 2\times 18 = 72$.

The grand total for the six cases appears to be $72 + 18 + 75 + 2 + 12 + 1 = 180$. The fact that it's a number with so many factors makes us suspect that there is a more efficient way to analyze this…

2013 Mu Alpha Theta National Convention
InterschoolA Test Solutions

4. **In the grid below, cells can be shaded to form a continuou path that forms no loops and never contains a 2x2 square so the cells labeled "e" border (including the cell itself) an even number of shaded cells and the cells labeled "o" border an o number of shaded cells. What is the maximum possible nu of shaded cells?**

e	e	o	e	o	e	e	o
o	e	o	e	e	e	e	o
o	o	o	o	o	e	e	e
e	e	o	e	e	o	o	o
o	e	e	o	o	o	o	e
o	o	o	e	o	o	o	e
o	e	o	o	e	e	o	o
o	e	e	e	e	e	o	e

I think the only way to break into this problem is to start in the corners.

The upper left cell is an "e", so of the four cells in that corner either 0, 2, or 4 are shaded. It cannot be all four cells, so really either 0 or 2 of those four cells are shaded.

The next cell to the right is also "e", so in addition to the 0 or 2 cells in the corner, either 0 or 2 of the top two cells of the next column must also be shaded. This is important, as it means that those two cells are the same as one another: either both shaded or both unshaded. You can label them both "A", reserving "B" for cells you determine must be the opposite of these cells.

The cell below the corner cell is an "o", so in addition to the 0 or 2 cells in the corner, exactly one of the leftmost two cells of the next row must also be shaded. This is important, as it means that those two cells are different from one another: one is shaded and one is unshaded. You can label one "C" and the other "D".

The cell diagonally adjacent to the corner cell (part a) is an "e", so in addition to 0 or 2 of the corner cells, 0 or 2 of the cells to the right, and 1 of the cells below, there must be one other shaded cell: the third cell of the third column must be shaded!

Similar analyses of the other corners yield three more certainties and various options in the third rows and columns from the edges. After this, you can start connecting all of your A's, B's, C's, etc. across the entire puzzle, ending up with just a few pairs of potential shaded/unshaded sets which you can case test for contradictions.

In the end, there are two possible solutions as shown below. The one on the left has more shaded cells, giving an answer of 38.

e	e	o	e	o	e	e	o
o	e	o	e	e	e	e	o
o	o	o	o	o	e	e	e
e	e	o	e	e	o	o	o
o	e	e	o	o	o	o	e
o	o	o	e	o	o	o	e
o	e	o	o	e	e	o	o
o	e	e	e	e	e	o	e

e	e	o	e	o	e	e	o
o	e	o	e	e	e	e	o
o	o	o	o	o	e	e	e
e	e	o	e	e	o	o	o
o	e	e	o	o	o	o	e
o	o	o	e	o	o	o	e
o	e	o	o	e	e	o	o
o	e	e	e	e	e	o	e

2013 Mu Alpha Theta National Convention
InterschoolA Test Solutions

5. a-d. List four fractions with positive numerators and denominators less than 1000 that approximate to .13579.

If the denominator were 1000, the numerator would be just less than 136, so we only need to consider numerators up to 135.

If we divide a candidate numerator (e.g. 111) by 0.13579, we will get a number (817.4) that may be "close" to the denominator (817 or 818) of a fraction that approximates to this (neither 111/817 or 111/818 does). This is quickly done in the accompanying Excel workbook for all numerators up to 135, with conditional formatting highlighting the answers $\frac{96}{707}$, $\frac{107}{788}$, $\frac{118}{869}$, and $\frac{129}{950}$.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

a. An ordered collection of positive integers can be augmented by having additional numbers inserted into the order in any position (but the order of the original elements is preserved). If the collections {1, 1, 2, 0, 1, 0, 2}, {1, 2, 1, 0, 1, 2}, and {1, 2, 2, 1, 1, 0, 2, 2} are all the result after a particular collection was augmented in three different ways, what are the largest collections that might have been the original collection?

The original collection can have at most five elements. If so, the three augmentations added 2, 1, and 3 elements, respectively.

When one element was added, there was one 0, three 1's, and two 2's, so that the original collection must have had at least two 1's and at least one 2. When two elements were added, there were two 0's, three 1's, and two 2's, so this augmentation definitely added at least one of the 0's. When three elements were added, there was one 0, three 1's, and four 2's, so that at least two 2's were added.

At this point, we can be sure that the original collection had 0-1 0's, 2-3 1's, and 1-2 2's. This is at most six elements, so really, we can think of it as having a 0, three 1's, and two 2's, but then one is removed.

Let's assume it started with no 0's, so that it was three 1's and a two 2's in some order. It would have had to be 12112 by the smallest augmentation, but this wouldn't work for the middle augmentation, so the original collection must have contained a 0, which is the one 0 of the smallest augmentation.

If the original collection had just one 2, then it was either 11012 or 12101 by the smallest augmentation. The middle augmentation only allows the former, but the largest augmentation does not allow it, so the original collection must have had two 2's and only two 1's.

Thus, the original collection must have been 21012, 12012, or 12102 by the smallest augmentation. The middle augmentation only allows 12012 and 12102, and the largest augmentation only allows 12102, so this is the answer.

b. An ordered collection of positive integers can be reduced by having numbers removed from any position (but the order of the other elements is preserved). If the collections {1, 1, 2, 0, 1, 0, 2}, {1, 2, 1, 0, 1, 2}, and {1, 2, 2, 1, 1, 0, 2, 2} are all the result after a particular collection was reduced in three different ways, what are the smallest collections that might have been the original collection?

The collection in question must have at least two 0's, at least three 1's, and at least four 2's. Starting with the largest collection, 12211022, it would obviously have to have a 0 added to be able to be reduced to the medium collection. Even with that, however, it still does not work. Adding a 1 up front gives 112211022, and adding a 0 after the first 2 and before the third 1 would give 1120211022, 1122011022, or 1122101022, any of which can be reduced to 1120102. Can any of these be reduced to 121012? Yes, the third option, 1122101022, can be reduced to 121012, making this our answer.

c. An ordered collection of positive integers can be mixed by removing a number from any position (but the order of the other elements is preserved) and inserting that number in another position (still preserving the order of the other elements). The collections {2, 2, 1, 2, 2, 1, 0, 1}, {2, 1, 1, 0, 2, 2, 1, 2}, and {1, 2, 2, 1, 1, 0, 2, 2} are all the result after a particular collection was mixed in three different ways. If the maximum number of total mixings needed to create these collections is a minimum, what collections might have been the original collection?

The first and third collections both have double 2's. The first collection can be turned into the third in three mixings: move a 1 from the back to the front, move a 1 from the back to the middle, move a 0 from the back to the middle. The intermediate collections would be 12212210&12211220. We could have done these mixings in another order, or even switched which 1's we moved to which positions, which could produce these intermediate collections: 12212210&12210221, 12212201&12211220, 12212201&12210221, 22112210&12211220, 22112210&22110221, 22112201&12211220, 22112201&22110221, and 22102211&12210221, 22102211&22110221. This is only 8 distinct collections (12212210, 12210221, 12212201, 12211220, 22112210, 22110221, 22112201, and 22102211), each of which requires a total of 3 mixings to get to the first and third collections. Adding the first and third collections to this list gives 10 collections. To turn any of these into the second collection requires that we split up a double 2, probably the first one. 22110221 is really close to the second collection, requiring just one mixing (sending the first 2 to the end), for a total of four mixings, which seems to be the fewest total mixings.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

7. When you look at a five-by-five city grid of building lots from the north, the tallest buildings you see in each column of buildings are 5, 4, 4, 5, and 2 tall. When you look from the east, the tallest buildings you see in each row of buildings are 5, 5, 4, 3,and 5 tall.

a. What is the smallest possible total height of all buildings (including potential zeros for empty lots)?

There are at least three 5's, at least two 4's, and at least one 3 & 2, for a total of 28.

b. How many arrangements are possible for this minimum total?

	5	4	4	5	2
5	5	4	4	5	2
5	5	4	4	5	2
4		4	4		2
3	3	3	3	3	2
5	5	4	4	5	2

In the figure to the right, there are only six places that can contain a 5 (the intersection of a 5-row and a 5-column), but there must be one 5 in each of the 5-rows and at least one in each of the 5-columns, so there are only $2^3 - 2\times1^3 = 8 - 2 = 6$ ways to arrange the 5's. There are 8 positions that could contain a 4, but there must be one in each 4-column and at least one in the 4-row, so there are only $4^2 - 3^2 = 16 - 9 = 7$ ways to arrange the 4's. There are 4 positions that could contain a 3 and five that could contain a 2, for a total of $6\times7\times4\times5 = 840$ ways to arrange the buildings.

c. What is the *largest* possible total height?

In the same figure, there are at most six 5's, then ten 4's (the two blanks could be 4's), then four 3's, and finally five 2's, for a total of 92.

d. How many arrangements are possible for this maximum total?

There is only one way to arrange the buildings in this case.

e. How many arrangements are possible if there are exactly three buildings of each height (1, 2, 3, 4, and 5)?

Again, there are six ways to arrange the 5's.

For each arrangement, there are $4\times4 - 3 = 13$ positions that could be 4's (the unused 5's, the blanks, and the 4's in the figure), but there must be at least one in each 4-column and the 4-row. There are $3\times3\times2 = 18$ ways to arrange the 4's without using the two cells that overlap rows & columns, $2\times3\times9 = 54$ ways using the overlap exactly once, and 11 ways using both overlapping cells, for a total of $18 + 54 + 11 = 83$ arrangements. In the case where we used the overlap exactly once, we found $2\times3\times5 = 30$ ways using three different columns, $2\times3\times3 = 18$ ways using the column of overlap twice, and $2\times3 = 6$ ways using the other potential-overlap column twice.

For each arrangement of 5's & 4's, there are $4\times5 - 2\times3 = 20 - 6 = 14$ positions that could be 3',s, but there must be at least one in the 3-row. This gives $14c3 - 10c3 = 14\times13\times2 - 10\times3\times4 = 364 - 120 = 244$ total ways.

For each arrangement of 5's, 4's, and 3's, there are $5\times5-3\times3 \; = 25-9 = 16$ positions that could be 2's, but there must be at least one in the 2-column. This gives $16c3 - 11c3 = 16\times5\times7 - 11\times5\times3 = 560 - 165 = 395$ total ways.

For each arrangement of 5's, 4's, 3's, and 2's, there are $5\times5-4\times3 \; = 25-12 = 13$ positions that could be 1's. This gives $13c3 = 13\times2\times11 = 286$ total ways.

These combine for an answer of $6\times83\times244\times395\times286 = 13727210640$.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

8. A positive integer is written …DCBA (if there were 26 digits, the first digit would be called Z), and has the property that the digit A is the remainder when the number is divided by 3, B is the remainder when the number is divided by 4, C is the remainder when the number is divided by 5, etc. What is the *n*th-smallest such number for each value of *n* below?

a. 1

The digit A can be 0, 1, or 2. For a number of any size, all of the other digits must sum to a multiple of 3. As a single digit, it obviously cannot be 0, but 1 is indeed the remainder when 1 is divided by 3, so the answer is 1.

b. 2

Similarly, this answer is 2.

c. 3

At this point, we must have at least a two-digit number. B can be 0, 1, 2, or 3, while A can be 0, 1, or 2. For a number of any size, the leading digits don't matter for B's rule. If B is 0, A must be 0. If B is 1, A cannot be 0, 1, or 2, so B cannot be 1. If B is 2, A must be 2. If B is 3, A must be 1. Thus, any number must end in 00, 22, or 31. Of these, only 31 works as a two-digit number, because A requires that the sum of all the other digits is a multiple of 3.

d. 4

Examining three-digit numbers, C must be the same as A, because A will always be the remainder when the number is divided by 5, so all numbers must end in 000, 222, or 131. Of these, none can be a three-digit solution. Examining four-digit numbers, D is the remainder when the number is divided by 6. This will either be A or $A + 3$, but because of even/odd parity it can only be A, so all numbers must end in 0000, 2222, or 1131. 2222 satisfies the condition for A, that all digits other than A sum to a multiple of 3, so it is the only four-digit answer.

e. 5

In general, E can be 0-6. For five-digit numbers (this will change for longer numbers), to satisfy A we only need to consider 30000, 60000, 32222, 62222, 11131, and 41131. 62222 and 11131 give the right remainder when divided by 7, so this part's answer is 11131.

f. 6

As we learned in part e, the answer is 62222.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

g. 7

Considering six-digit numbers, for numbers of any size, F is determined by A, B, and C, so that numbers must end 0E0000, 6E2222, 3E1131. To satisfy A, the only six-digit numbers to consider are 602222, 632222, 662222, 311131, and 341131. Of these, only 632222 works when you investigate the remainder when dividing by 7, so that is the answer to this part.

h. 8

For seven-digit numbers, for the digit G to be the remainder when the number is divided by 9, the sum of the other numbers must be a multiple of 9, which means we must consider numbers of the form G000000, G642222, G301131. To satisfy A, the only seven-digit numbers to consider are 3000000, 6000000, 2642222, 5642222, 8642222, 1301131, 4301131, and 7301131. Testing for divisibility by 7, we find that none of these have the proper remainder.

Examining eight-digit numbers, H must be the same as A, so we must consider 0G0E0000, 2G6E2222, 1G3E1131. For an eight-digit number, the digits other than G must sum of a multiple of 9, so we need only consider 2G622222. For A's constraint, we can narrow it down to 22622222, 25622222, and 28622222. Of these, none has the appropriate remainder when dividing by 7.

Nine-digit numbers must be of the form I0G0E0000, I2G6E2222, and I1G3E1131. For the first case, divisibility by 9 requires that $I + E$ is a multiple of 9, which could be 9 (10 ways) or 18 (1 way), divisibility by 3 requires that G is 0, 3, or 6, and divisibility by 11 requires that $I + G + E$ is a multiple of 11, which is impossible given the other constraints. For the second case, $I + E$ must be 2 or 11, so that G must be 2, 5, or 8 in either case, and $I + G + E$ must be 8 or 19, which is possible if $I + E = 11$ and $G = 8$ (5 ways). For the third case, $I + E$ must be 8 or 17, G must be 1, 4, or 7, and $I + G + E$ must be 6 or 17, which is not possible given the other constraints. Thus, we must investigate the numbers 528662222, 628652222, 728642222, 828632222, and 928622222. Checking their divisibility by 7, none of these work, either.

Ultimately, we didn't have the time or inclination to look beyond this, and none of the submissions had larger numbers that worked out, so we're dropping parts h-l.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

9. Digits are written repeatedly in the squares of a sheet of graph paper as shown in each part below. If you can use scissors to make any vertical or horizontal cut you like along the gridlines (you do not have to cut the entire width or height of the paper at once), what is the greatest number of distinguishable rectangular scraps of paper you can create? Assume that all numbers are distinctive enough that you cannot confuse them for one another, even when rotated. The back of the paper is blank.

a.

12312
12312
12312
12312
12312

For this and all parts of the problem, there are a fixed number of digits ($5\times 5 = 25$ in this case), so we'd like to have scraps with fewer digits than more. In this particular case, we have five fewer 3's than we have 1's or 2's, which could affect the answer.

There are three one-digit scraps, which means three digits have been used, there are 22 digits remaining, and one of our five 3's has been used.

There are three horizontal two-digit scraps and three vertical two-digit scraps, using another twelve digits, leaving ten digits and no more 3's. Perhaps not using one or more of these pieces with 3's on them would help us later.

There are 3 horizontal (none usable) and 3 vertical (at most 2 usable) three-digit scraps. Using what works with the earlier choices, we use another six digits, leaving four digits (and still no 3's).

There are 3 square four-digit scraps (1 usable), which we can use, for a total of $3 + 3 + 3 + 2 + 1 = 12$ pieces. Choosing not to use some smaller pieces with 3's on them doesn't gain us anything, unfortunately.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

b.

123412341
234123412
341234123
412341234
123412341
234123412
341234123
412341234
123412341
234123412

Smaller pieces are still better. Note that we have 81 digits and just one extra 1, so concerns about the number of 1's used probably won't influence our answer.

There are four one-digit scraps using a total of four digits, leaving 77. There are a total of eight two-digit scraps using a total of sixteen digits, leaving 61. There are a total of eight three-digit scraps using a total of 24 digits, leaving 37. There are two different shapes of four-digit scraps, totaling to 12 scraps using a total of 48 digits, which is 11 too many. Thus, we can only use nine of the four-digit scraps, which gives us an answer of 29 scraps and a lot of flexibility on how we might cut that many from the grid.

c.

1264446264
1235535535
1634462644
3254531113
5234562262
3614361135
1216521264
1212561235
1636341634
3254543254

There are 100 digits, with an extra 1, 2, 3, & 4, so concerns about extra digits probably won't influence our answer.

There are six one-digit scraps using a total of six digits, leaving 94. There is the potential to have 36 different two-digit scraps in each direction, for a total of 72 scraps and 144 digits, which is way more than we can use. If enough of these combinations exist in the grid (it appears most do) and they can be arranged properly, the answer will be $6 + \frac{94}{2} = 6 + 47 = 53$.

d.

0123456789012345678901234567890123456

… (22 more rows, for a total of 23) …

There are $23 \times 37 = 851$ digits, with 23 extra 0's-6's, which will definitely complicate things.

There are ten one-digit scraps using ten digits total, leaving 841. There twenty two-digit scraps using forty digits total, leaving 801. There are twenty three-digit scraps using sixty digits total, leaving 741. There are thirty four-digit scraps using a total of 120 digits, leaving 621. There are twenty five-digit scraps using a total of 100 digits, leaving 521. There are a total of forty six-digit scraps using a total of 240 digits, leaving 281. There are a total of twenty seven-digit scraps using a total of 140 digits, leaving 141.

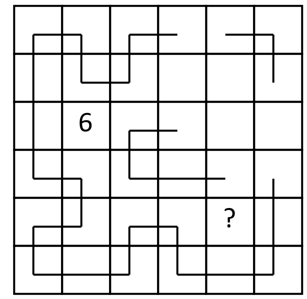
At this point, we will not be able to use all of the possible 8-digit scraps. In fact, we will only be able to use $\frac{141}{8} \sim 17$ 8-digit scraps, which would leave 5 spare digits. Note that the one- to seven-digit scraps use all the digits equally, so that they use $\frac{710}{10} = 71$ of each digit. There are only $23 \times 3 = 69$ 7's, 8's, and 9's, so we've used two too many of each. If we leave out the six-digit piece using 789 twice, then we use 159 one- to seven-digit scraps with a total of $710 - 6 = 704$ digits, leaving $851 - 704 = 147$ spares, $23 - 2 = 21$ each of 0's-6's.

If we use 8-digit scraps to use up these extra 0's-6's, we can use seven 1x8 vertical strips and six 2x4 vertical strips, for a total of 172 scraps leaving nine 0's & 6's and five 1's-5's. Next we can use the 4x2 pieces from 0-3 and 3-6, for 174 scraps leaving seven 0's & 6's, three 1's, 2's, 4's, & 5's, and one 3. Finally, we can use two nine-digit squares from 0-2 and 4-6, for 176 scraps leaving four 0's & 6's and one 3.

We've actually made a diagram with 175 pieces (we couldn't place the vertical 22), but we believe the answer is 176.

2013 Mu Alpha Theta National Convention InterschoolA Test Solutions

10. Each of the grids below can be filled with a closed path of line segments connecting the centers of the cells horizontally or vertically and not branching at all. The path will pass through all cells except those containing numbers or ?'s, and the cells (up to 8) around each cell with a number will contain that number of turns (e.g. a 6 will have 6 turns in the cells surrounding it). For each grid, what is the maximum number of turns in a solution? The example grid to the right has the parts of the path that can be deduced for certain, and there are four possible ways the path could be completed, with the maximum number of turns being 24.



a.

							3
	5			5			
				3	4		
	6						
						5	
		5					

b.

				4					
								6	
	7								
				5					
									?
		8					7		
					6				

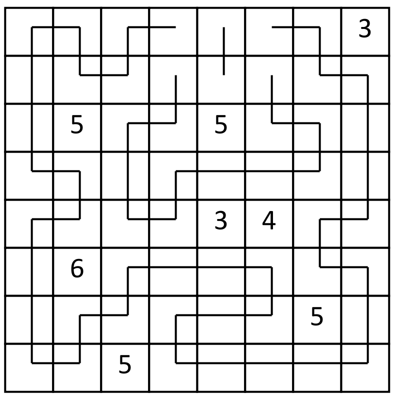
c. In this part, none of the ?'s is a 5.

			?								
5				5				8			5
				5							
		?				5		5			
?											
			5				?		?		?
	?										
								?			

2013 Mu Alpha Theta National Convention
InterschoolA Test Solutions

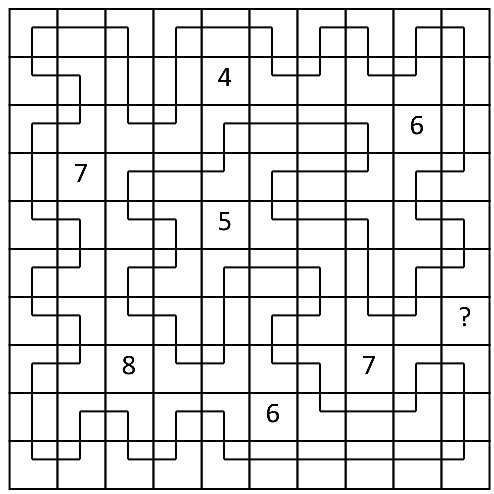
Worrying about the basic layout first, in corners and number-dense regions, seems to be the best way to start. When you're trying to connect loose ends in multi-solution regions, parity issues can quickly decrease the number of solutions to consider.

a.



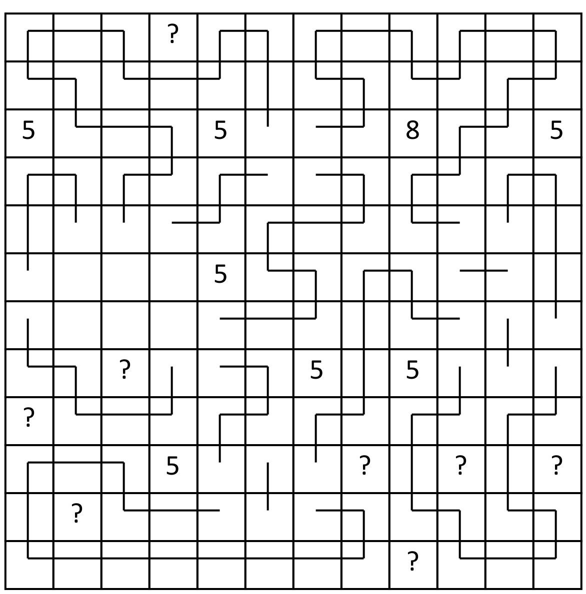
Two solutions.
4+4+4+4+6+4+4+4=34 certain corners
+4 either way 34+4=38 total corners

b.



One solution.
8+6+6+6+8+8+8+8+6+6=70

c. In this part, none of the ?'s is a 5.



This problem turned out to be impossible, as the lowest ? must have 5 turns around it.