Mu Alpha Theta National Convention 2013
Mu Division – Gemini Test Solutions

1. **(C → 4)** How many x-values on the interval $[0, 2\pi]$ satisfy the equation $\cos x = \sin(2x)$?

$$\cos x = \sin(2x)$$
$$\cos x = 2\sin x \cos x$$
$$\cos x - 2\sin x \cos x = 0$$
$$\cos x(1 - 2\sin x) = 0$$
$$\cos x = 0 \quad 1 - 2\sin x = 0 \rightarrow \ \sin x = \frac{1}{2}$$
$$\text{x} = \frac{\pi}{2}, \frac{3\pi}{2} \text{ and x} = \frac{\pi}{6}, \frac{5\pi}{6}$$

2. **(D → 316)** A = the amplitude of $y = 3sinx + 4cosx$.
   B = the volume of a cone with slant height 5 and radius 3.
   C = 75^2
   D = the second smallest positive perfect number
   What is $\frac{C}{25} + 3A + \frac{4B}{\pi}$ + D?

   The amplitude of a function with two different trig functions is the square of their amplitudes added together and square rooted.
   A = $3^2 + 4^2 = 25$
   A = $\sqrt{25} = 5$

   Height = $\sqrt{5^2 - 3^2}$ = 4
   B = $\pi(3^2)(4) = 36\pi$

   C = 5625

   Positive perfect numbers: 6, 28, 496
   D = 28

   $\frac{C}{25} + 3A + \frac{4B}{\pi}$ + D = 316

3. **(A → 13)** What is the local maximum value of the function $f(x) = 4x^3 + 3x^2 - 6x + 8$ on the interval $-2 < x < 2$?

$$\text{f}'(\text{x}) = 12x^2 + 6x - 6 = 0$$
$$\text{x} = \frac{1}{2}, -1$$
$$\text{f}\left(\frac{1}{2}\right) = 6.25, f(-1) = 13$$
$$\text{Max} = 13$$

4. **(B→ 286)** Chloe has 16 identical candy bars. How many ways can she distribute these among herself and three other friends if she must keep at least three for herself and her other friends must each receive at least one? Assume that the candy bars are identical and the children are not.

Chloe keeps 3 for herself and gives 3 away, 1 to each of her friends so that everyone meets their requirements. This leaves 10 candy bars to distribute randomly to 4 people. Use pigeon hole principle.

$$\binom{10+4-1}{4-1}$$
$$\binom{13}{3} = 286$$

5. **(D→13)** Solve $13^{2013} \equiv x \bmod 17$ where x is a positive whole number less than 17.

$$13^2 = 169 \equiv -1 \bmod 17$$
$$(13^2)^{1006}(13) \equiv x \bmod 17$$
$$1(13) \equiv x \bmod 17$$
$$x \equiv 13$$

6. **(A→240)** Find the coefficient of x^4 in the expansion of $(2x + \frac{2}{x} + 3)^5$

$$\text{Coefficient} = (2x)^4(3)\frac{5!}{4!\,1!}$$
$$\text{Coefficient} = 240$$

7. **(C→$\frac{1}{11}$)** Let g(x) be the inverse of $f(x) = 3x^2 - 7x + 12$ where $x > 2$. Find $g'(18)$.

When x = 3, y = 18. In the inverse, we get x = 18, y = 3.

Inverse function → $x = 3y^2 - 7y + 12$

$$1 = 6yy' - 7y'$$
$$1 = 6(3)y' - 7y' = 18y' - 7y'$$
$$1 = 11y'$$
$$y' = \frac{1}{11}$$

8. **(E→520)** Find the area of the region bounded by the x-axis and $f(x) = 2x^3 - 8x$ on the interval $0 \le x \le 6$.

$$2x^3 - 8x = 0$$
$$x = 0, 2$$
$$\int_0^2 8x - 2x^3 + \int_2^6 2x^3 - 8x = 520$$

9. **(B → January 28, 2020)** If today is July 25, 2013, what day will it be 2378 days from today? For example, 1 day from today would be July 26, 2013, 2 days would be July 27, 2013, etc.

10. **(C→$\frac{68\pi}{3}$)** Find the volume of the solid obtained by rotating the area bounded by the function f(x) = $x^4 - 2x + 3$ and the x-axis on the interval $0 \le x \le 2$ about the y-axis.

$$2\pi \int_0^2 x(x^4 - 2x + 3)dx = \frac{68\pi}{3}$$

11. **(A→e^3)** Compute $\sum_{n=0}^{\infty} \frac{3^n}{n!}$

$$\sum_{n=0}^{\infty} e^x = \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{3^n}{n!} = e^3$$

12. **(D→61)** The value of sin(3) using the first three terms of the Maclaurin series of sin x is equal to $\frac{m}{n}$. Compute the value of m + n, where m and n are relatively prime.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\sin 3 = 3 - \frac{3^3}{3!} + \frac{3^5}{5!}$$

$$\sin 3 = \frac{21}{40}$$

21+40=61

13. **(A→$\frac{2}{3}$)** Find the volume of a tetrahedron with vertices A(1, -3, 5), B(4, 0, 3), C(2, -3, 4), and D(3, 1, 3)
X = B – A = (3, 3, -2)
Y = C – A = (1, 0, -1)
Z = D – A = (2, 4, -2)

$$\frac{1}{6}\det \begin{matrix} 3 & 3 & -2 \\ 1 & 0 & -1 \\ 2 & 4 & -2 \end{matrix}$$

$$\frac{1}{6}(4) = \frac{2}{3}$$

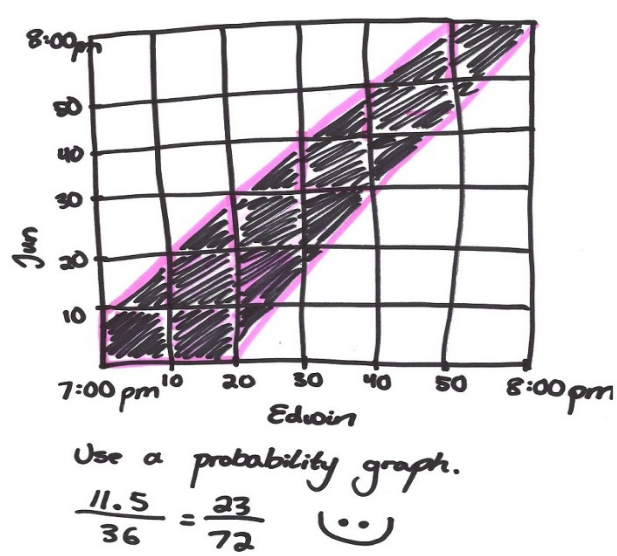
14. **(A→11π)** Find the area of the polar curve r = 3 + 2cosθ.

$$A = \frac{1}{2}\int r^2$$

$$A = \frac{1}{2}\int (3 + 2\cos\theta)^2$$

$$A = 11\pi$$

15. **(D→$\frac{23}{72}$)** Jun and Edwin are trying to meet up for dinner sometime between 7:00 pm and 8:00 pm. However, both of them forget to tell the other party what time to meet up. Jun will arrive sometime between those two hours and wait ten minutes for Edwin and leave. Edwin will arrive sometime between those hours and wait a third of the time left till 8:00 pm and leave. What is the probability that they will meet and end up eating dinner together?



16. **(E→0)** What is $\lim_{x\to 0}\frac{\cos x - 1}{x}$?

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

$$(-1)\lim_{x\to 0}\frac{1-\cos x}{x} = 0$$

17. **(E → $\frac{\sqrt{2}(\pi+2)}{4}$)** Find the derivative of $f(x) = x^2 \arcsin x$ at $x = \frac{\sqrt{2}}{2}$.

$$f'(x) = 2x\arcsin x + \frac{x^2}{\sqrt{1 - x^2}}$$

$$f'\left(\frac{\sqrt{2}}{2}\right) = 2\left(\frac{\sqrt{2}}{2}\right)\arcsin\left(\frac{\sqrt{2}}{2}\right) + \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}}$$

$$f'\left(\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}(\pi + 2)}{4}$$

18. **(B→$\frac{-14}{25}$)** Let Mln|x| + Nln|x + 5| + C equal the indefinite integral $\int \frac{2-x}{x^2+5x}$dx where C, M, and N are constants. Find MN.

Use partial fractions.

$$\frac{A}{x} + \frac{B}{x+5} = \frac{2-x}{x^2+5}$$

$$A = \frac{2}{5}, B = \frac{-7}{5}$$

$$\int \frac{\frac{2}{5}}{x} + \frac{\frac{-7}{5}}{x+5} dx$$

$$\frac{2}{5}\ln|x| - \frac{7}{5}|x+5| + C$$

$$M = \frac{2}{5}, N = \frac{-7}{5}$$

$$MN = \frac{-14}{25}$$

19. **(A → Vigesimal)** What numeral system did the Mayan calendar use?

20. **(D→22)** What is the smallest number of vertices a planar graph can have if it has at least 60 edges?

Derived from planar graphs Euler's formula.

$$E \leq 3V - 6$$

$$60 \leq 3V - 6$$

$$V \geq 22$$

21. **(E→$\frac{7}{2}$)** Evaluate $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7i}{n^2}$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{7i}{n^2} = \lim_{n\to\infty} \frac{7(1+2+3+4\ldots)}{n^2} = \lim_{n\to\infty} \frac{7n(n+1)}{2n^2} = \frac{7}{2}$$

22. **(C→265)** Six people placed their cell phones on a table. Each of them picked up a cell phone at random. In how many ways could this have been done so that no person picks up their own cell phone?

How to derange six items: $6!(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}) = 265$

23. **(A→17)** Let $0.\overline{mn}_5 = \frac{5}{6}$. Find $2^m + 3^n$

$$\overline{mn} = \frac{5}{6}$$

$$\frac{5}{6} = 4\left(\frac{1}{5^1}\right) + 1\left(\frac{1}{5^2}\right) + 4\left(\frac{1}{5^3}\right) + 1\left(\frac{1}{5^4}\right) \dots$$

$$\frac{5}{6} = 0.404040 \dots = 0.\overline{40}$$

$$2^4 + 3^0 = 17$$

24. **(E→$\frac{63\pi}{16}$)** Find the volume of the region bounded by the curves $f(x) = 2x^2 - 3x + 2$ and $g(x) = 4x - 3$ rotated about the y-axis.

$$A = \int_1^{\frac{5}{2}} 4x - 3 - (2x^2 - 3x + 2)\,dx = \frac{9}{8}$$

$$\bar{x} = \frac{1}{\frac{9}{8}} \int_1^{\frac{5}{2}} x(4x - 3 - (2x^2 - 3x + 2)dx = \frac{7}{4}$$

$$\text{Theorem of Pappus} = 2\pi RA = 2\pi\left(\frac{7}{4}\right)\left(\frac{9}{8}\right) = \frac{63}{16}\pi$$

25. **(E→323)** A rabbit is at the origin of the number line and moves according to the following rule: the rabbit jumps to the closest point with a greatest integer coordinate that is a multiple of 3, or to the closest point with a greatest integer coordinate that is a multiple of 17. A move sequence is a sequence of coordinates which correspond to valid moves, beginning with 0, and ending with 54. For example, 0, 3, 17, 34, 51, 54 is a move sequence. How many move sequences are possible for the rabbit?

Use a table representing the number of ways to reach a certain number. Add up the number of ways you can reach a certain number by adding all the ways you can get to the previous numbers that will get to that number.

0	3	6	9	12	15	17	18	21	24	27	30	33	34	36	39	42	45	48	51	54
1	1	1	1	1	1	6	7	7	7	7	7	7	48	55	55	55	55	55	323	323

26. **(B→32)** Find the smallest integer n (not 0) such that $1^2 + 2^2 + 3^2 + 4^2 + \cdots + n^2$ is a multiple of 250. Let 100x + 10y + z = n, where x, y, and z are positive integers. Find x^2y^3z when z > y > x.

$$\text{S} = \frac{n(n+1)(2n+1)}{6} \textit{ is a multiple of } 250 \textit{ if } n(n+1)(2n+1) \textit{ is a multiple of } 1500.$$

$1500 = 2^2 3^1 5^3$ *so* $4, 3, 125 | n(n+1)(2n+1)$. Since 2n+1 is always odd, and only one of n and n+1 is even, either n, n+1$\equiv$ 0 mod 4. Therefore, n $\equiv$ 0, 3 mod 4.

If $n \equiv 0 \bmod 3$, then $3|n$. If $n \equiv 1 \bmod 3$, then $3|2n+1$. If $n \equiv 2 \bmod 3$, then $3|k+1$. Therefore, there are no restrictions on k in mod 3.
Only one of n, n+1, 2n+1 will be divisible by 5. So either n, n+1, 2n+1 $\equiv 0 \bmod 125$. Therefore, $n \equiv 0, 124, 62 \bmod 125$.
From Chinese Remainder Theorem, $n \equiv 0, 124, 187, 312, 375, 499 \bmod 500$, The smallest integer for n that is not 0 is 124.
$1^2 2^3 4 = 32$

27. **(C→4)** Let $y = \frac{-e^{Lx}}{M} + Ce^{Nx}$ be the solution to the differential equation $y' - 2y = e^{-x}$, where C, L, and M are constants. Find L + M + N.

Multiply each side by e^{-2x}

$$e^{-2x}y' - 2e^{-2x}y = e^{-3x}$$
$$(e^{-2x}y)' = e^{-3x}$$
$$e^{-2x}y = \frac{-e^{-3x}}{3} + C$$
$$y = \frac{-e^{-x}}{3} + Ce^{2x}$$
$$L + M + N = 4$$

28. **(B→**$\ln 2 + \frac{\pi^2}{6}$) What is the sum of this infinite sequence?

$$1 + 1 - \frac{1}{2} + \frac{1}{4} + \frac{1}{3} + \frac{1}{9} - \frac{1}{4} + \frac{1}{16} \dots$$

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots = \ln 2$
$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \dots = \frac{\pi^2}{6}$

29. **(E→**$\frac{\pi}{e^2}$**)** Evaluate $\int_1^\infty \frac{4}{e^{x+1}+e^{3-x}} dx$

$$u = x - 1, du = dx$$
$$4\int_1^\infty \frac{dx}{e^{x+1} + e^{3-x}} = 4\int_0^\infty \frac{du}{e^{u+2} + e^{2-u}}$$
$$\frac{4}{e^2}\int_0^\infty \frac{du}{e^u + e^{-u}} du$$
$$4\int_0^\infty \frac{e^u}{(e^u)^2 + 1} du$$
$$v = e^u, dv = e^u$$
$$\frac{4}{e^2}\int_1^\infty \frac{dv}{v^2 + 1} = \frac{1}{e^2} \arctan v \textit{ from } 1 \textit{ to } \infty$$

$$\frac{4}{e^2}\left(\frac{\pi}{2}-\frac{\pi}{4}\right) = \frac{\pi}{e^2}$$

30. **(A→1247)** Let S be the sum of all numbers of the form $\frac{m}{n}$ where m and n are relatively prime positive divisors of 2000. What is the greatest integer that does not exceed $\frac{S}{4}$?

$$2000 = 2^4 5^3$$

$\frac{m}{n}$ can be expressed in form $2^x 5^y$, where $-4 \le x \le 4, -3 \le y \le 3$.

So every number of the form $\frac{m}{n}$ will be expressed once in the expression

$$S = (2^{-4} + 2^{-3} + 2^{-2} + 2^{-1} + 2^0 + 2^1 + 2^2 + 2^3 + 2^4)(5^{-3} + 5^{-2} + 5^{-1} + 5^0 + 5^1 + 5^2 + 5^3) = \left(\frac{2^{-4}(2^9-1)}{2-1}\right)\left(\frac{5^{-3}(5^7-1)}{5-1}\right)$$

$$S = \left(\frac{2^{-4}(2^9-1)}{2-1}\right)\left(\frac{5^{-3}(5^7-1)}{5-1}\right) = 4990 + \frac{341}{2000}$$

$$\left\lfloor \frac{S}{4} \right\rfloor = 1247$$

Tiebreaker

The curve $y = 2x^2 - 3$ is revolved around the y-axis to form a bowl. If water flows into the bowl at a rate of 3 cubic units per minute, how fast is the depth of the bowl changing in terms of π when the depth of the bowl is 6 units?

$$\Delta V = \pi x^2 dy$$

$$V = \pi \int_{-3}^{h} \frac{y+3}{2} dy$$

$$\frac{dV}{dt} = \frac{dV}{dh}\cdot\frac{dh}{dt} = \pi\frac{h+3}{2}\cdot\frac{dh}{dt}$$

$$3 = \pi\frac{6+9}{2}\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{2}{5\pi}$$