Number Theory
Open, Round 1
Test #101

1. Write your 6-digit ID# in the I.D. NUMBER grid, left-justified, and bubble. Check that each column has only one number darkened.
2. In the EXAM NO. grid, write the 3-digit Test # on this test cover and bubble.
3. In the Name blank, print your name; in the Subject blank, print the name of the test; in the Date blank, print your school name (no abbreviations).
4. Scoring for this test is 5 times the number correct + the number omitted.
5. You may not sit adjacent to anyone from your school.
6. TURN OFF ALL CELL PHONES OR OTHER PORTABLE ELECTRONIC DEVICES NOW.
7. No calculators may be used on this test.
8. Any inappropriate behavior or any form of cheating will lead to a ban of the student and/or school from future national conventions, disqualification of the student and/or school from this convention, at the discretion of the Mu Alpha Theta Governing Council.
9. If a student believes a test item is defective, select "E) NOTA" and file a Dispute Form explaining why.
10. If a problem has multiple correct answers, any of those answers will be counted as correct. Do not select "E) NOTA" in that instance.
11. Unless a question asks for an approximation or a rounded answer, give the exact answer.

Note: For all questions, answer "(E) NOTA" means none of the above answers is correct.

1. A *prime-like number* is a positive integer greater than 1 that is not divisible by 2, 3, or 5. What is the largest integer that cannot be expressed as the sum of two or more prime-like numbers?

   (A) 16 (B) 23 (C) 31 (D) 59 (E) NOTA

2. Define a *squarable rectangle* as a rectangle that can be partitioned into identical squares with integer side lengths of 2 or greater. How many distinct squarable rectangles are there with sides less than or equal to 10? Note: A 4×2 rectangle is distinct from a 2×4 rectangle.

   (A) 15 (B) 28 (C) 38 (D) 45 (E) NOTA

3. What fraction of the first 100! terms of the Fibonacci sequence (having both first and second terms equal to 1) are divisible by 7?

   (A) $\frac{1}{8}$ (B) $\frac{1}{7}$ (C) $\frac{1}{6}$ (D) $\frac{1}{5}$ (E) NOTA

4. How many positive factors of 12 are quadratic residues mod 47?

   (A) 2 (B) 3 (C) 4 (D) 6 (E) NOTA

5. For how many three-digit positive integers is the sum of the digits greater than the product of the digits?

   (A) 171 (B) 196 (C) 199 (D) 201 (E) NOTA

6. What is the remainder when a googolplex $\left(10^{\left(10^{100}\right)}\right)$ is divided by 23?

   (A) 6 (B) 13 (C) 16 (D) 20 (E) NOTA

7. How many zeros does 727! end in when written in base 45?

   (A) 105 (B) 180 (C) 235 (D) 360 (E) NOTA

8. How many unordered triples of positive primes $\{p, q, r\}$ satisfy $p + q + r = 50$?

   (A) 4 (B) 5 (C) 6 (D) 7 (E) NOTA

9. What is the smallest positive value of x satisfying the following system of congruences?

$$x \equiv 3 \pmod 7$$
$$x \equiv 4 \pmod{11}$$
$$x \equiv 8 \pmod{13}$$

(A) 213 (B) 346 (C) 675 (D) 983 (E) NOTA

10. Which one of the following is a primitive root of 13?

(A) 3 (B) 5 (C) 9 (D) 11 (E) NOTA

11. Let $x = 3^5 2^4$. How many positive divisors of x^2 are less than or equal to x, but do not divide x?

(A) 10 (B) 20 (C) 30 (D) 40 (E) NOTA

12. Find the number of unordered quadruples of positive integers $\{a, b, c, d\}$ that satisfy

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} = 1$$

(A) 6 (B) 10 (C) 12 (D) 14 (E) NOTA

13. For how many integer values of n is $n! + 3$ a perfect square?

(A) 2 (B) 3 (C) 4 (D) Infinitely many (E) NOTA

14. Which of the following numbers is deficient?

(A) 70 (B) 104 (C) 272 (D) 315 (E) NOTA

15. What is the product of the positive divisors of $2^3 7^4$?

(A) $2^6 7^8$ (B) $2^{18} 7^{24}$ (C) $2^{30} 7^{40}$ (D) $2^{60} 7^{80}$ (E) NOTA

16. Suppose x and y are integers such that $13x + 22y = 45$. What is the smallest possible value of $|x - y|$?

(A) 10 (B) 46 (C) 74 (D) 108 (E) NOTA

17. What are the last 2 digits of $\sum_{n=0}^{100} n!$ when expressed in the octal number system?

(A) 16_8 (B) 16_8 (C) 16_8 (D) 16_8 (E) NOTA

18. Which one of the following is composite?

(A) 127 (B) 257 (C) 1447 (D) 2047 (E) NOTA

19. What is the product of the GCD and LCM of 70 and 98?

(A) 3,430 (B) 6,860 (C) 13,720 (D) 17,150 (E) NOTA

20. A function f is surjective over the integers mod 31 if for every positive integer y less than 31 there exists some integer x such that $f(x) \equiv y \pmod{31}$. Which of the following functions is not surjective over the integers mod 31?

(A) $f(x) = x + 3$ (B) $f(x) = -5x$

(C) $f(x) = 2^x$ (D) $f(x) = x^{-1}$ (modular inverse) (E) NOTA

21. Convert 3,218 to base 7.

(A) $11{,}436_7$ (B) $11{,}623_7$ (C) $12{,}056_7$ (D) $12{,}245_7$ (E) NOTA

22. The four digit number $X45Y_{12}$ is divisible by 143_{10}. Evaluate $X + Y$ in base 10.

(A) 7 (B) 13 (C) 14 (D) 20 (E) NOTA

23. Find the 15th smallest positive integer with exactly 4 positive divisors.

(A) 26 (B) 39 (C) 51 (D) 62 (E) NOTA

24. Order the following from smallest to largest:
A = The number of positive primes less than 1,000,000.
B = The number of perfect squares less than 1,000,000.
C = The number of square-free positive integers less than 1,000,000.

(A) A, B, C (B) B, A, C (C) B, C, A (D) C, A, B (E) NOTA

25. Find the sum of all integers x between 1 and 12, inclusive, satisfying

$$x^2 + 6x - 7 \equiv 0 (\text{mod } 12).$$

(A) 6 (B) 12 (C) 24 (D) 36 (E) NOTA

26. In the following correctly-calculated multiplication problem, the prime digits have been replaced by the letter P and the perfect square digits have been replaced by the letter S. Find the sum of the digits of the product:

$$\begin{array}{r} PS \\ \times\, SP \\ \hline SPP \\ SPS \\ \hline SPSP \end{array}$$

(A) 9 (B) 12 (C) 18 (D) 22 (E) NOTA

27. What is the second smallest positive perfect number?

(A) 1 (B) 6 (C) 20 (D) 28 (E) NOTA

28. How many four-digit positive integers are there such that when they are subtracted from their reversals, the result is a power of 3?

(A) 48 (B) 55 (C) 62 (D) 63 (E) NOTA

29. What is the tens digit of $54{,}321^{9{,}876}$?

(A) 2 (B) 4 (C) 6 (D) 8 (E) NOTA

30. What is the largest odd square-free integer n such that $\varphi(n) = 1{,}200$? Note that $\varphi(n)$ is the Euler Totient Function, the number of positive integers less than n that are relatively prime to n.

(A) 1,201 (B) 1,271 (C) 1,705 (D) 5,115 (E) NOTA