Mu Alpha Theta Nationals 2013
Number Theory Test Open Division
Solution Sketches

1) List the next few primes, 7, 11, 13, 17, 19, 23. Add them to each other to discover you cannot achieve a sum of 23, but can achieve 24 through 30. By adding 7 you can achieve all values larger than 23.

2) A rectangle is squarable if it's sides are coprime. Count the number of such ordered pairs directly.

3) Evaluate each of the first 7 mod 7 and it becomes clear the every 7^{th} one divides 7.

4) Check that 2 and 3 are quadratic residues using Legendre symbols. Since they are all of their multiples are. 1 trivially is. Thus all roots of 12 are.

5) There are
+ 90 numbers with 0 as a second digit
+ 90 numbers with 0 as a third digit
- 9 with last two digits 0s
+ 27 numbers with 2 or more 1s
- 2 for 110 and 101
+3 that are permutations of 122
For a total of 199

6) Use Euler's theorem twice dividing out by 2 when necessary.

7) Count the number of 3s (divided by 2) and 5s by repeatedly dividing remainders. Use the lesser value.

8) One prime must be 2. The others can be {5, 43}, {7, 41}, {11 37}, {17, 31}, and {19, 29}

9) Basic modular arithmetic

10) Raise numbers to 4^{th} and 6^{th} powers mod 13. See which one is not 1 for either.

11) x^2 has 99 factors. 50 of them are less than or equal to x. Since x has 30 factors that leaves 20 remaining values.

12) Count them (or know Egyptian fractions)

13) The only values that work less than 6 are 0, 1, 3. For 6 and higher it is always congruent to 3 mod 9 and not a perfect square.

14) Check directly

15) Let x = $2^3 7^4$. x has 20 factors, which can be divided into 10 pairs each multiplying to x. thus the answer is x^{10}

16) Find a possible pair using Extended Euclidean Algorithm, then add subtract by 22 and 13 as necessary to move them nearest to 0

17) Check up through 7. Afterwards the sum is 0 in modulo 8.

18) This is the first Mersenne false prime

19) Multiply 70 and 98

20) The others are surjective. 2^x only maps to 1, 2, 4, 8, and 16.

21) Compute directly

22) By the base 12 divisibility rule for 13 (same as 11 base 10), $X + 1 \equiv Y (\text{mod } 13)$. By the base 12 divisibility rule for 11 (same as 9 base 10), $X + Y \equiv 2 (\text{mod } 11)$. (6, 7) is the only solution that satisfies.

23) List pseudo primes and perfect cubes. The 15th value is 46.

24) There are 1,000 squares, ~78,500 (1,000,000/6 ln(10)) primes, and ~600,000 (60% of 1,000,000) square free

25) We have 12 | (x – 1)(x + 7). The roots are 1 and 5 (-7) and 11, and 7 also work.

26) The only solution is:

$$\begin{array}{r} 51 \\ \times\ 93 \\ \hline 153 \\ 459 \\ \hline 4743 \end{array}$$

27) 6 is the smallest. 28 is next.

28) Let the larger number be ABCD (Note D>0). We have ABCD - DBCA = 9 * (111 * (A - D) + 10 * (B – C)). If A – D = 1 and B – C = -3, then the difference is 729. There are 56 ways for this to happen. If A – D = 3 and B – C = -9, then the difference is 2187. There are 6 ways for this to happen.

29) 21 has order 5 mod 100. Thus it raised to anything ending in 6 will end in a 21.

30) Take the factors of 1200 and see which ones are 1 less than primes. Find the set of these that multiply to 1200 and their corresponding prime has the highest product. Smaller primes (especially 3) are better because the +1 counts for more (3/2 instead of 13/12). The largest such product is 2013.