Mu Alpha Theta National Convention: San Diego 2013
Theta Matrices and Determinants Topic Test – Solutions

1. B
2. B
3. A
4. C
5. D
6. E
7. D
8. C
9. D
10. D
11. C
12. D
13. A
14. E
15. A
16. A
17. B
18. B
19. A
20. B
21. E
22. D
23. C

24. B

25. B

26. C

27. D

28. D

29. E

30. B

1. **(B)**. Perform operations on each individual entry
$b + 2(2b) = 25 \Rightarrow b = 5$
$-3 + 2a = -7 \Rightarrow a = -2$
$a + b = 3.$

2. **(B)**. Expanding by minors yields
$$\begin{vmatrix} 3 & 4 & 11 \\ 15 & -6 & -10 \\ -12 & 2 & 7 \end{vmatrix} = 3\begin{vmatrix} -6 & -10 \\ 2 & 7 \end{vmatrix} - 4\begin{vmatrix} 15 & -10 \\ -12 & 7 \end{vmatrix} + 11\begin{vmatrix} 15 & -6 \\ -12 & 2 \end{vmatrix}$$
$$= 3(-42 - (-20)) - 4(65 + 120) + 11(30 - 72)$$
$$= -468.$$

3. **(A)**. We have
$$(A+B)^2 + B^{-1} = A^2 + AB + BA + B^2 + 6B^{-1}$$
$$= \begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}$$
$$+ 6\begin{bmatrix} 2 & 3 \\ 2 & 6 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} 45 & 16 \\ -16 & -3 \end{bmatrix} + \begin{bmatrix} 18 & 33 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 8 & 7 \\ 2 & 10 \end{bmatrix} + \begin{bmatrix} 10 & 24 \\ 16 & 42 \end{bmatrix} + 6(1/6)\begin{bmatrix} 6 & -3 \\ -2 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 87 & 77 \\ -2 & 51 \end{bmatrix}.$$

4. **(C)**. Simply multiplying yields
$$\begin{bmatrix} (6)(4)+(1)(0) & (6)(3)+(1)(-8) & (6)(-7)+(1)(9) \\ (1)(4)+(0)(0) & (1)(3)+(0)(-8) & (1)(-7)+(0)(9) \\ (4)(4)+(-1)(0) & (4)(3)+(-1)(-8) & (4)(-7)+(-1)(9) \end{bmatrix} = \begin{bmatrix} 24 & 10 & -33 \\ 4 & 3 & -7 \\ 16 & 20 & -37 \end{bmatrix}$$
The sum of the entries is 0.

5. **(D)**. The adjoint of a matrix is the transpose of the matrix where each entry is replaced by its cofactor. The cofactor $C_{ij} = (-1)^{i+j}M_{ij}$, where M_{ij} is the minor expansion along

the i^{th} row and j^{th} column.
The matrix of cofactors is
$$\begin{bmatrix} (-1)^{(1+1)}((4)(-1)-(2)(-2)) & (-1)^{1+2}((3)(-1)-(5)(-2)) & (-1)^{1+3}((3)(2)-(5)(4)) \\ (-1)^{2+1}((0)(-1)-(2)(1)) & (-1)^{2+2}((1)(-1)-(5)(1)) & (-1)^{2+3}((1)(2)-(0)(5)) \\ (-1)^{3+1}((0)(-2)-(4)(1)) & (-1)^{3+2}((1)(-2)-(3)(1)) & (-1)^{3+3}((1)(4)-(3)(0)) \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -7 & -14 \\ 2 & -6 & -2 \\ -4 & 5 & 4 \end{bmatrix}$$
The transpose of the cofactor matrix is
$$\begin{bmatrix} 0 & 2 & -4 \\ -7 & -6 & 5 \\ -14 & -2 & 4 \end{bmatrix}.$$

6. **(E)**. The cofactor $C_{ij} = (-1)^{i+j}M_{ij}$. So $C_{32} = (-1)^{3+2}M_{32} = (-1)(-5) = 5$ and $M_{14} = 22$. Hence, $C_{32}M_{14} = 110$.

7. **(D)**. The sum of eigenvalues is equal to the trace of the matrix. $Tr(A) = 18$. The product of eigenvalues is equal to the determinant of the matrix. $Det(A) = 100$. So let a and b equal the other two eigenvalues. $1 + 10 + a + b = 18$ and $1 \cdot 10 \cdot a \cdot b = 100$. So $a + b = 7$ and $ab = 10$. By inspection, the other two eigenvalues are 5 and 2. Hence, $5^2 + 2^2 = 29$.

8. **(C)**. We can simply figure out which vector is an eigenvector by multiplying each vector by the matrix. If the product is a scalar multiple of the vector, then it is an eigenvector of the matrix. Try $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$:
$\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. It is a scalar multiple of $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$.

9. **(D)**. $Det(A) = 1001$. $Det(2A) = 2^3 \cdot 1001$ and $Det(A^{-1}) = \frac{1}{1001}$.
So $Det(2A) - 7007 \cdot \frac{1}{1001} = 8001$.

10. **(D)**. A singular matrix has a determinant of zero.
$\begin{vmatrix} 5 & -2 & 1 \\ 1 & 0 & 3 \\ -1 & 1 & x \end{vmatrix} = -15 + 2x + 6 + 1$. So $-8 + 2x = 0 \Rightarrow x = 4$.

11. **(C)**. By definition, an orthogonal matrix is one where the inverse is equal to its transpose. So $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ is an orthogonal matrix, while the others are not.

12. **(D)**. To find the rank of a matrix, first convert the matrix to reduced row echelon form by row operations. Then count the number of non-zero rows and that is the rank. The matrix in question reduces to $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ so it has a rank of 3.

13. **(A)**. A 3 x 7 matrix can have rank values of 0, 1, 2, or 3, so the sum of possible rank values is 6.

14. **(E)**. By the standard formula, we can find the area of the triangle as follows:
$$A = \frac{1}{2}\begin{vmatrix} 3 & -8 & 1 \\ 6 & 8 & 1 \\ -1 & -6 & 1 \end{vmatrix} = 35.$$

15. **(A)**. The trace is the sum of entries along the diagonal. So the trace of the matrix is $9 + 3 + 1 + \frac{1}{3} + ... + 9 \cdot \frac{1}{3}^{19} = \frac{9(1-\frac{1}{3}^{20})}{2/3} = \frac{27}{2}(1 - \frac{1}{3}^{20}) = \frac{1}{2}(27 - 3^{-20} \cdot 3^3) = \frac{1}{2}(27 - 3^{-17})$.

16. **(A)**. To find an inverse of a matrix, augment it with the identity matrix on the right and row reduce until the identity matrix is on the left side.
After using various row operations on the augmented matrix, $\left[\begin{array}{ccc|ccc} 1 & 4 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 3 & 5 & 3 & 0 & 0 & 1 \end{array}\right]$ becomes $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & 3 & -3 & -1 \\ 0 & 0 & 1 & -6 & 7 & 2 \end{array}\right]$. Hence, the inverse matrix is $\begin{bmatrix} 1 & -2 & 0 \\ 3 & -3 & -1 \\ -6 & 7 & 2 \end{bmatrix}$.

17. **(B)**. To find the determinant of this matrix, first simplify each element. By simple log rules, the simplified matrix is $\begin{bmatrix} -4 & 6 \\ 1 & -4 \end{bmatrix}$. The determinant is $16 - 6 = 10$.

18. **(B)**. By inspection, $w = 0, x = 1, y = 0$, and $z = 0$ so $(wy + xz)^5 = 0^5 = 0$.

19. **(A)**. By the standard formula, the determinant that represents the equation of a circle containing the points $(1, -4), (3, -6)$, and $(3, -2)$ is $\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ 17 & 1 & -4 & 1 \\ 45 & 3 & -6 & 1 \\ 13 & 3 & -2 & 1 \end{vmatrix}$. Setting the determinant equal to 0 yields $8x^2 - 48x + 8y^2 + 64y + 168 = 0$. The equation can be rewritten as $(x - 3)^2 + (y + 4)^2 = 4$, which is a circle.

20. **(B)**. A is a Markov matrix so it is helpful to find the eigenvectors of the matrix. We can find eigenvalues by setting the following matrix equal to 0. $\begin{vmatrix} 0.5 - \lambda & 0.25 \\ 0.5 & 0.75 - \lambda \end{vmatrix} = (0.5 - \lambda)(0.75 - \lambda) - \frac{1}{4} \cdot \frac{1}{2} = 0$. After solving for λ, we get $\lambda = 1$ and $\lambda = 0.25$ The corresponding eigenvectors are $V_1 = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$ and $V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Express X_0 in terms of the two eigenvectors, which is $2V_1 - 1V_2$. So $X_1 = A(2V_1 - V_2) = 2AV_1 - AV_2 = 2V_1 - \frac{1}{4}V_2$. As X_2, X_3, and so on are calculated the second term approaches zero and we are left with $2AV_1$ so $X_{5,000,000,000}$ is equal to $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The magnitude of the vector is $\sqrt{1^2 + 2^2} = \sqrt{5}$.

21. **(E)**. Look at the elements of each matrix modulo 2. Since every matrix has odd elements along the main diagonal and even elements elsewhere, the matrix in modulo 2 is equivalent to the 4×4 identity matrix, resulting in a determinant of 1. Hence, none of the matrices are singular, because zero is an even number.

22. **(D)**. Let us first look at the first few powers of the matrix to find a pattern: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^1 = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^2 = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^3 = \begin{bmatrix} 36 & 28 \\ 28 & 36 \end{bmatrix}$, $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^4 = \begin{bmatrix} 136 & 120 \\ 120 & 136 \end{bmatrix}$, and so on. Upon inspection, a pattern emerges: $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^n = \begin{bmatrix} 2^{2n-1} + 2^{n-1} & 2^{2n-1} - 2^{n-1} \\ 2^{2n-1} - 2^{n-1} & 2^{2n-1} + 2^{n-1} \end{bmatrix}$.

23. **(C)**. First, place 0s and 1s in any order in the 3×3 matrix in the upper-left hand corner. There are 2^9 ways to do this. Then place a 0 or 1, depending on what is necessary in the first three entries in the fourth row and then the entries down the fourth column. This can be done in exactly one way.

24. **(B)**. Since A^3 is equal to the zero matrix, the determinant of A^3 is 0, hence the determinant of A is also 0, so A is singular. The rest of the matrices are not singular, and we can explicitly construct their inverses: $(I - A)^{-1} = I + A + A^2$, $(I + A)^{-1} = I - A + A^2$, and $(I + A + \frac{1}{2}A^2)^{-1} = I - A + \frac{1}{2}A^2$.

25. **(B)**. We have $(A^2 + B^2)(A - B) = A^3 - A^2B + B^2A - B^3$, which simplifies to the zero matrix, based on the information given in the problem. Therefore, either the determinant of $A^2 + B^2$ or $A - B$ is 0. Since A and B are different matrices, we know it's the former.

26. **(C)**. ABC is not well-defined because A has 50 columns and B has 20 rows. AB^TC is not well-defined because B^T has 20 columns and C has 30 rows. CA^2B is not well-defined because C has 20 columns and A^2 has 50 rows. AB^TC^T is well-defined because A has 50 columns and B^T has 50 rows. Also, B^T has 20 columns and C^T has 20 rows.

27. **(D)**. The determinant of A^2 is always a non-negative square number so answer choices A, B, and C are eliminated because they all have negative determinants. D is a perfect square because $\begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}^2 = \begin{bmatrix} 11 & 7 \\ 14 & 18 \end{bmatrix}$.

28. **(D)**. Row operations can be performed on $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to make it look like $\begin{bmatrix} 3c & 3d \\ 4a - c & 4b - d \end{bmatrix}$. Since the second row was multiplied by 3 and the first row was multiplied by 4, the determinant is multiplied by 12. Switching rows multiplies the determinant by -1. Subtracting a multiple of a row from another row does not change the determinant. All in all, the determinant is multiplied by -12. So the value of $|\frac{m}{n}| = \frac{12}{1}$. So $m + n = 13$.

29. **(E)**. Simplifying the expression, we get $e^{14} = \ln(e^e \cdot a - 0)$. Exponentiate both sides to get $e^{e^{14}} = e^e \cdot a$. Solving for a yields $a = e^{e^{14} - e}$.

30. **(B)**. The determinant of $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = aei + bfg + cdh - ceg - bdi - afh$. Choosing 9, 8, and 7 for a, e, and i to make a large positive term leads to minimizing fh, cg, and bd to reduce the magnitude of the negative terms. Choosing 5, 4, and 6 for c, d, and h works to make another large positive term and minimize the magnitude of the negative terms. 3, 1, and 2 are left for b, f, and g. The final matrix is $\begin{bmatrix} 9 & 3 & 5 \\ 4 & 8 & 1 \\ 2 & 6 & 7 \end{bmatrix}$, which has a determinant of 412.