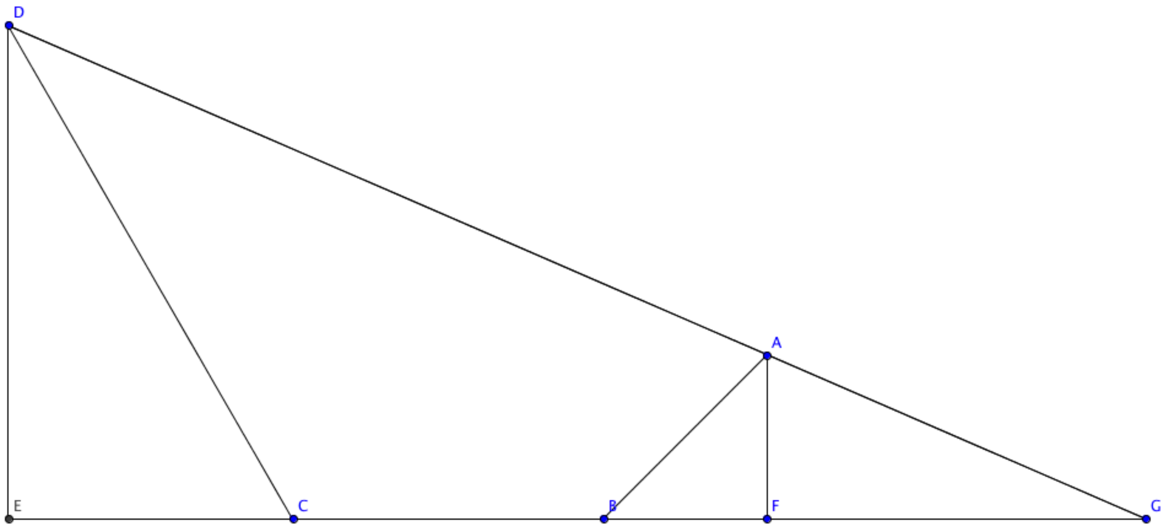
Question	Solution
P1.	We have $12^2+36^2=12^2(1+3^2)=144(10)=\mathbf{1440}$.
P2.	The area is $(6)(4)/2 = \mathbf{12}$.
P3.	We have $x=102-4=\mathbf{98}$.
P4.	One-sixth of the area is $(2(3)^{1/4})^2\sqrt{3}/4=3$, so the total area is $(3)(6)=\mathbf{18}$.
P5.	We have $\frac{A}{B}+C-D=\frac{1440}{12}+98-18=\mathbf{200}$.
1.	If $x=3$, then $\|3-2\|<\|3-6\|$, or $1<3$, which is true. However, if $x=4$, then $\|4-2\|<\|4-6\|$, or $2<2$, which is false. The largest integer solution is therefore $x=\mathbf{3}$.
2.	Represent the number as $ABCD$; the given information leads to the equations $B=3+A$, $C=2B-5=2(3+A)-5=2A+1$, $D=2A+1$, and $A+B+C+D=A+(3+A)+2(2A+1)=23$. Thus, $A=3$ and the number is 3677, which is a prime number. So the answer is **3677**.
3.	Let the triangle have side lengths a, b, and c, with c as the length of the hypotenuse. We then have $a+b+c=12+8\sqrt{3}$ and $a^2+b^2+c^2=294$. By the Pythagorean Theorem, $a^2+b^2+c^2=c^2+c^2=2c^2=294$, or $c=7\sqrt{3}$. Therefore, $a+b=12+\sqrt{3}$ and $a^2+b^2=147$. Square both sides of the first equation to obtain $a^2+2ab+b^2=147+24\sqrt{3}$. Thus, $2ab=24\sqrt{3}$, making the area equal to $ab/2 = 24\sqrt{3}/4=\mathbf{6\sqrt{3}}$.
4.	We have $4^x-4^{x-1}=4^x-\left(\frac{1}{4}\right)4^x=\left(\frac{3}{4}\right)4^x=24$, so $4^x=2^{2x}=32=2^5$, so $2x=5$. Therefore, $(2x)^{2x}=5^5=\mathbf{3125}$.

5.	We have $\frac{B-D}{A}+C^2=\frac{3677-3125}{3}+\left(6\sqrt{3}\right)^2=\mathbf{292}$.
6.	Since $2^{12}<5566<2^{13}$, the binary representation of 5566 is a 1 followed by 12 other digits for a total of **13** digits.
7.	The common difference of the sequence is $1-4=-3$, and so the number of terms is $\frac{-32-4}{-3}+1=\mathbf{13}$.
8.	There are 16 ways to choose one endpoint. After that, there are only $16-1-2=13$ possibilities for the other endpoint since adjacent vertices cannot be used. Since order does not matter, the number of diagonals is $(16)(13)/2=\mathbf{104}$.
9.	The relation can be expressed as $f=Kg^2h$ for some K, which, in this case, is equal to $K=\frac{f}{g^2h}=\frac{128}{4^2\times 2}=4$. The answer is then $f=4(3^2)(6)=\mathbf{216}$.
10.	We have $\log(D-C-B+1)^A=\log(216-104-13+1)^{13}=\log 100^{13}=\mathbf{26}$.
11.	If $L(x)=mx+b$, then $I(x)=\frac{x-b}{m}$. So we have $mx+b=\frac{4(x-b)}{m}+3$, or $mx+b=\frac{4x}{m}+3-\frac{4b}{m}$. Set corresponding coefficients equal to each other to obtain the equations $m=4/m$ and $b=3-\frac{4b}{m}$. Since the slope is positive, $m=2$. Plug this into the second equation to get $b=3-\frac{4b}{2}=3-2b$, or $b=1$. Thus, $L(10)=2(10)+1=\mathbf{21}$.
12.	For every $k\times k$ square whose vertices are in S with sides parallel to the coordinate axes, there are $k-1$ squares that can be inscribed inside the square. Thus, each $k\times k$ square produces a total of $k-1+1=k$ squares including the one with sides parallel to the axes. For a given k, there are $(5-k)^2$ squares in S. The total number of squares is then $\sum_{k=1}^{4}k(5-k)^2=\mathbf{50}$.

13.	By Euler's Theorem, units digits of bases relatively prime to 10 have a period of 4. Since 2013 has a remainder of 1 upon division by 4, the units digit of $4 \cdot 3^{2013}$ has the same units digit as $4 \cdot 3^1 = 12$, so **2**.
14.	For positive x, $g^{-1}(x) = x^2 - 2$ (this is easy to check just by plugging it into g and seeing that the identity function is obtained), so $g^{-1}(5) = 23$ and $f(g^{-1}(5)) = f(23) =$ $\mathbf{65}$.
15.	We have $BD^C + A = (50)(65)^2 + 21 = \mathbf{211271}$.
16.	By inspection, $x = \mathbf{1}$.
17.	The left-hand side of the equation can be interpreted as the number of ways to choose three things from $3n$ distinct objects, or $\binom{3n}{3}$. Since $\binom{9}{3} = 84$, we have $3n = \mathbf{9}$.
18.	By the properties of determinants, $\det(2A)\det(3B) = 2^2 \det(A)\, 3^3 \det(B) =$ $(4)(2)(27)(-3) = \mathbf{-648}$.
19.	By inspection, $x = 1$ is a solution. Via Synthetic Division, $x^3 - 7x + 6 = (x-1)(x-2)(x+3)$, so $x = 2$ is the other positive root. The desired product is therefore **2**.
20.	We have $B^{-1}C(2A - D) = (9^{-1})(-648)(2(1) - 2) = \mathbf{0}$.
21.	Hexagonal numbers have a second-level common difference of $6 - 2 = 4$ and octagonal numbers have a second-level common difference of $8 - 2 = 6$. Thus, the hexagonal numbers are $1, 6, 15, 28, \ldots$ and the octagonal numbers are $1, 8, 21, 40, \ldots$. The answer is $15 + 40 = \mathbf{55}$.
22.	The maximum occurs when $b = 19$. Since $360 = 18(19) + 18$, $f(19) = 18^2 = \mathbf{324}$.

23.	Since $x^2 + 4x + 9 = (x + 2)^2 + 5$, and the square of any real number is nonnegative, the minimum value is **5**.
24.	If $\log_k 3 \leq 4$, then $k^4 \geq 3$, or $k^8 \geq 9$, making the answer **9**.
25.	We have $\frac{A}{C} + \frac{B}{D^2} = \frac{55}{5} + \frac{324}{81} = \mathbf{15}$.
26.	Notice that vectors **a**, **b**, and **c** are mutually orthogonal to each other. Let $\mathbf{d} = [-6, -17, 6]$. We have $c_1 = \frac{\mathbf{d \cdot a}}{\mathbf{a \cdot a}} = \frac{12}{2} = 6$, $c_2 = \frac{\mathbf{d \cdot b}}{\mathbf{b \cdot b}} = \frac{-68}{34} = -2$, and $c_3 = \frac{\mathbf{d \cdot c}}{\mathbf{c \cdot c}} = \frac{51}{17} = 3$, so $c_1 c_2 c_3 = \mathbf{-36}$.
27.	The least common multiple of 3, 5, and 7 is 105. Based on the given congruences, $x + 2$ is a multiple of 3, 5, and 7. Thus, $x + 2 = 105n$ for integers n, or $x = 105n - 2$. The largest negative solution closest to 0 is $\mathbf{-2}$.
28.	Use as many 2s and 3s as possible to obtain as many factors that yield an increasing product. In particular, use as many 3s as possible more than 2s. The solution to $2x + 3y = 31$ with the largest possible value of y is $x = 2$ and $y = 9$. Thus, the largest possible product is $2^2 \times 3^9 = 78732$, and the sum of the digits is **27**.
29.	The distance between the centers of the circles, $(3, 12)$ and $(-4, -12)$, is $\sqrt{7^2 + 24^2} = 25$. The sum of the radii of the circles is $12 + 13 = 25$. Thus, the circles intersect at exactly one point, which has area **0**.
30.	We have $\|A\| + \|B\| + \|C\| + \|D\| = 36 + 2 + 27 + 0 = \mathbf{65}$.
31.	The coordinates of triangle POQ are $(0, 0)$, $(5, 0)$, and (x, y), where $x^2 + y^2 = 36$. The centroid of POQ is the average of the coordinates, or $\left(\frac{x+5}{3}, \frac{y}{3}\right)$. Suppose $\left(\frac{x+5}{3}, \frac{y}{3}\right) = (a, b)$ so that $\frac{x+5}{3} = a$ and $\frac{y}{3} = b$. Solving each equation for x and y, squaring both sides, and adding the equations, we arrive at $(3a - 5)^2 + (3b)^2 = 36$, or $\left(a - \frac{5}{3}\right)^2 + b^2 = 4$, an

	equation of a circle of radius 2. The desired area is $\boldsymbol{4\pi}$.
32.	Let x equal the number of black marbles. Consequently, x is also the number of white marbles. We have $\frac{\binom{x}{3}}{\binom{2x}{3}} = \frac{1}{12}$, yielding $x = 5$. Therefore, there are **10** marbles in the jar.
33.	The quickest way to solve this problem is guess-and-check. If $x = 14$ and $y = -6$, then $x + y = 8$ and $x^2 + y^2 = 232$. Thus, $\|x\| + \|y\| = 14 + 6 = \mathbf{20}$.
34.	Since $4096^2 = 4^{12}, 8^4 = 2^{12}, 81^3 = 3^{12}$, and $25^6 = 5^{12}$, the smallest element in D is 8^4, so $f(m) = f(8^4) = \mathbf{100}$.
35.	We have $A^{-1}C\pi \log_B D = (4\pi)^{-1}(20)\pi \log 100 = \mathbf{10}$.
36.	The equation is of the form $M = PDP^{-1}$, where D is a diagonal matrix. Therefore, we have $M^{10} = (PDP^{-1})^{10} = PD^{10}P^{-1}$. Notice that $D^{10} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}^{10} = \begin{bmatrix} (-1)^{10} & 0 \\ 0 & 1^{10} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the identity matrix I. Thus, $M^{10} = PD^{10}P^{-1} = PIP^{-1} = PP^{-1} = I$. The sum of the elements of the 2×2 identity matrix is **2**.
37.	The number is $\left(\frac{1}{4}\right)6868 + 68 = 1717 + 68 = \mathbf{1785}$.
38.	Since $135 = 3^3 \times 5$ and $84 = 2^2 \times 3 \times 7$, we have $\text{LCM}(135, 84) = 2^2 \times 3^3 \times 5 \times 7 = \mathbf{3780}$.
39.	The sum of the digits of $10^1 - 1$ is $9 \times 1 = 9$. The sum of the digits of $(10-1)(10^2 - 1) = 891$ is $9 \times 2 = 18$. Basically, adding another term in the product in accordance with the pattern creates a bunch of new digits whose sum is 9, with the number of "bundles" of digits whose sum is 9 equal to the number of 9s in the largest factor. Specifically, the sum of the digits of $(10-1)(10^2-1)(10^4-1)(10^8-1)$ is $9 \times 8 = \mathbf{72}$

	because $10^8 - 1$ has eight 9s in its base-10 representation.
40.	We have $(BCD)^{A-2} = (BCD)^{2-2} = \mathbf{1}$.
41.	 Starting with quadrilateral $ABCD$, draw auxiliary line segments to obtain the diagram above, where AF and DE are perpendicular to EG, which contains the points C, B, and F. Triangle ABF is a 45-45-90 triangle, so $AF = BF = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$. Triangle DCE is a 30-60-90 triangle, so $EC = 3$ and $DE = 3\sqrt{3}$. Triangles AFG and DGE are similar. Therefore, $\frac{AF}{DE} = \frac{FG}{EG}$, or $\frac{\sqrt{3}}{3\sqrt{3}} = \frac{FG}{3+5-\sqrt{3}+\sqrt{3}+FG} = \frac{FG}{8+FG}$, or $FG = 4$. Moreover, $AG = \sqrt{AF^2 + FG^2} = \sqrt{\sqrt{3}^2 + 4^2} = \sqrt{19}$. Based on the earlier equation, triangles AFG and DGE are in a 3-to-1 linear ratio, so $AD = 2 \times AG = \mathbf{2\sqrt{19}}$.
42.	If the roots are a and b, then $a + b = -8$ and $ab = -1$. Let $a^3 + b^3 = N$. Since $(a+b)^3 - (a^3 + b^3) = 3a^2b + 3ab^2 = 3ab(a+b)$, we have $(-8)^3 - N = 3(-1)(-8)$, or $-512 - N = 24$, making $N = \mathbf{-536}$.
43.	If $5x - 4 = 16$, then $10x - 8 = 32$, so $10x - 14 = 10x - 8 - 6 = 32 - 6 = \mathbf{26}$.

44.	The entry in the second row, second column of M^{-1} is the second row, second column of the transposed adjoint matrix of M, divided by $\|M\|$. The second row, second column cofactor of M is $(-1)^{2+2}\begin{vmatrix} -1 & -5 \\ 4 & 5 \end{vmatrix} = 15$. Thus, $\frac{15}{4x-47} = 15$, so $x = \mathbf{12}$.
45.	We have $\sqrt{\|10(A^2 - C) + B\|} + D = \sqrt{\left\|10\left(\left(2\sqrt{19}\right)^2 - 26\right) - 536\right\|} + 12 = \mathbf{18}$.
46.	The numbers being plugged into the function are the first five positive perfect numbers. Recall that even perfect numbers have the form $g(x) = 2^{x-1}(2^x - 1)$, where $2^x - 1$ is prime; by inspection, the five smallest positive values of x which makes this true are 2, 3, 5, 7, and 13. We have $f\left(g(x)\right) = \log_2\left(1 + \sqrt{8(2^{x-1}(2^x - 1)) + 1}\right) - 2 = x - 1$. Thus, the answer is $(2-1) + (3-1) + (5-1) + (7-1) + (13-1) = \mathbf{25}$.
47.	The common difference is $\frac{-24-(-40)}{3-1} = 8$. The terms given are multiples of 8 and thus repeated additions of 8 will ultimately yield a term of 0, and the term after that will be the first positive term, **8**.
48.	Suppose $(6x - 2y + z)^5 = a_1x^5 + a_2x^4y + \cdots + a_kz^5$. Notice that substituting 1 for all the variables has the effect of leaving only the coefficients intact, and moreover, in this process, is already being summed up. The answer is $(6 - 2 + 1)^5 = 5^5 = \mathbf{3125}$.
49.	There are three possible cases that are not necessarily disjoint: 1. 53AB 2. A53B 3. AB53 where A and B is any valid digit. Case 1 has $10 \times 10 = 100$ ways to occur, Case 2 has $9 \times 10 = 90$ ways to occur, and Case 3 has $9 \times 10 = 90$ ways to occur. Between these three cases, only Case 1 and Case 3 have a possibility of overlapping: the number 5353. By the Principle of Inclusion-Exclusion, the total number of possibilities is $100 + 90 +$

	$90 - 1 = \mathbf{279}$.
50.	We have $D\log_{B}\left(\frac{A+C}{25}+2\right) = 279\log_8\left(\frac{25+3125}{25}+2\right) = 279\log_8(128) = 279\left(\frac{7}{3}\right) = \mathbf{651}$.