Theta Sequences & Series – Solutions

2013 MAΘ National Convention

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1) $\sum_{n=2}^{2012} n = 2025077$ E

2) $P_5 = 11 \rightarrow \sum_{n=1}^{11} P_n = 2+3+5+7+11+13+17+19+23+29$
$+31$

$= 160$ C

3) $d = \frac{60-1}{5-1} = \frac{59}{4}$ C

4) $\sum_{l=1}^{10} l = 55$, regardless of index summation letter.

$55^4 = 9150625$ B

5) $\frac{25(25+1)(2\cdot 25+1)}{6} = 5525$ B

6) 1, 5, 12 22 B

4 7 10

3 3

7) 23_4, 24_5, 25_6

$\downarrow \quad \downarrow \quad \downarrow$

$11 \xrightarrow{+3} 14 \xrightarrow{+3} 17 \xrightarrow{+3} 20_{10} = 24_8$ C

8) $\prod_{n=1}^{100} i^n = i^{1+2+3+\dots+100} = i^{5050} = i^2 = -1$

B

9) $\frac{3}{4}$ 1 $\frac{4}{3}$ $\frac{16}{9}$ $\frac{64}{27}$ $\frac{781}{108}$ A

10)

Day	Chris	Kadie	Total	Cumulative
1	1	2	3.0	3
2	2.5	0	2.5	5.5
3	4.0	5	9.0	14.5
4	5.5	0	5.5	20
	⋮	⋮		⋮

It will take 35 days. E

11) If arithmetic

2nd worst

$$a + (a+d) + (a+2d) + (a+3d) = 30$$

worst

$$4a + 6d = 30$$

$$2a + 3d = 15$$

a	d	a+d
0	5	5
3	3	6
6	1	7

If geometric: 2nd worst

$$a + ar + ar^2 + ar^3 = 30$$

$$a(1-r^4) = 30(1-r)$$

worst

a and r have to be integers.

a	r	ar
2	2	4

$$5 + 6 + 7 + 4 = 22$$

A

12) $S = \frac{1}{5} + \frac{2}{25} + \frac{3}{125} + \ldots$

$$\frac{S}{5} = \frac{1}{25} + \frac{2}{125} + \ldots$$

$$\frac{4S}{5} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots = \frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{1}{4}$$

$$S = \frac{5}{16} = .3125$$

B

13) It takes an hour for 2 trains working backwards from point of impact to initial position at their given speeds. So the fly will travel 30 miles. D

14) $\underbrace{7,\ 8,\ 9,\ 8,}\ \underbrace{7,8,9,8,}\ \underbrace{7,8,9,8,}\ldots$

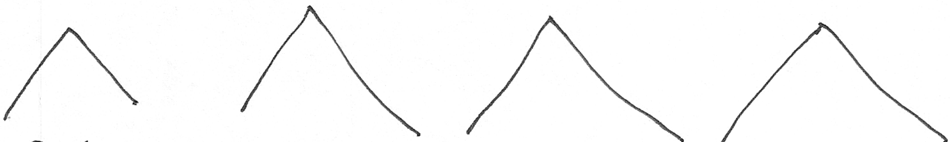
$2013 \equiv 1 \bmod 4$, so $a_{2013} = a_1 = 7$ B

15) $$\prod_{n=2}^{2}\frac{2+n}{n}+\prod_{n=2}^{3}\frac{3+n}{n}+\prod_{n=2}^{4}\frac{4+n}{n}+\prod_{n=2}^{5}\frac{5+n}{n}$$

$$\left(\frac{2+2}{2}\right)+\left(\frac{5}{2}\right)\left(\frac{6}{3}\right)+\left(\frac{6}{2}\right)\left(\frac{7}{3}\right)\left(\frac{8}{4}\right)+\left(\frac{7}{2}\right)\left(\frac{8}{3}\right)\left(\frac{9}{4}\right)\left(\frac{10}{5}\right)$$

$2 + 5 + 14 + 42 = 63$ A

16) D



2 terms 4 terms 6 terms 8 terms ....

a_{2013} will occur in the 45th "mountain"

a_{1980} a_{1982} a_{1983} .....

$1,\ 2,\ 3,\ \ldots\ 45,\ 45,\ 44,\ \ldots,\ 1$

$2013-1980=33$, so $a_{2013}=33$

17) HARMONIC D

18) Make all the numbers equal to 3.
Then average of their squares is $3^2 = 9$. E

19) $C = \left(\frac{2013^{2013}(2013^{2013}+1)}{2} \right)^2$, a perfect square. A perfect square has an odd number of positive divisors. E

20) Doesn't matter of series starts at $n = 2013$.
$|r| < 1$ for convergence. D

21) $(2x)^2 = (x-1)(5x+3) \longrightarrow x = \cancel{-1}, 3$ A

will have negative terms.

22) $.\overline{36} = \frac{36}{99}$

$.0\overline{36} = \frac{36}{990}$

$4.1\overline{36} = 4.1 + \frac{36}{990} = \frac{41}{10} + \frac{36}{990} = \frac{91}{22}$ B

$m + n = 113$

23) $r=2$

$6(2)^{n-1}=768 \rightarrow n-1=7$

$n=8$

A

24) $a_{2013} = S_{2013} - S_{2012}$

$= 4(2013)(2012) - 4(2012)(2011)$

$= 4(2012)(2013-2011) = 16096$

E

25) $\frac{a}{1-r} = 2013$

$\frac{a^2}{1-r^2} = (2013)(33)$

$\left(\frac{a}{1-r}\right)\left(\frac{a}{1+r}\right) = (2013)(33)$

$(\cancel{2013})\left(\frac{a}{1+r}\right) = (\cancel{2013})(33)$

$\frac{a}{1+r} = 33$

$\frac{r+1}{1-r} = 61 \rightarrow r = \frac{30}{31}$

C

26)

$$3y\sum_{n=1}^{6} n + \sum_{n=1}^{6} 2 = 3414$$

$$3y(21) + (6)(2) = 3414$$

$$y = 54 \qquad \text{A}$$

27)

$$\binom{13}{2}(2a)^2(-b)^{11}$$

$$(78)(4)(-1)^{11}a^2b^{11} = -312a^2b^{11} \qquad \text{D}$$

28)

First 100 Fibonacci numbers have ~~6~~7 odd terms and 33 even terms. To obtain a sum that's odd, a and b have to be different parity.

$$\frac{\binom{33}{1}\binom{67}{1}}{\binom{100}{2}} = \frac{67}{150} \qquad \text{C}$$

29) $x_1 = 2^a$, $r = 2^d$, $x_n = 2^a \cdot (2^d)^{n-1} = 2^{a+(n-1)d}$

$$\sum_{n=1}^{10} \log_2 x_n = \sum_{n=1}^{10} (a + (n-1)d) = 10a + 45d = 500$$

$$\downarrow$$

$$2a + 9d = 100$$

$(a,d) \in \{ (5,10), (14,8), (23,6), (32,4), (41,2) \}$

Consider $\log_2 \left(\sum_{n=1}^{10} x_n \right) = \log_2 \left(\frac{2^a((2^d)^{10}-1)}{2^d - 1} \right)$

$$\approx \log_2 \left(\frac{2^a \cdot 2^{10d}}{2^d} \right)$$

$$= \log_2 2^{a+9d} = a + 9d$$

We have $90 < a + 9d < 100$ only when $(a,d) = (5,10)$

so $\log_2 x_{20} = a + 19d = 5 + 19(10) = 195$ C

30) a_n = # of n-letter words w/ even # of As & Bs

b_n = " " " " w/ " " " " & odd # of Bs.

c_n = # of n-letter words w/ both odd # of A's and Bs.

- $a_n + b_n + c_n = 4^n$
- $a_{n+1} = 2a_n + b_n$
- $b_{n+1} = 2 \times 4^n$

$\longrightarrow a_{n+1} = 2a_n + 2 \cdot 4^{n-1}$, $a_1 = 2$ so $a_4 = 72$ C